Polynomial $f(R)$ Palatini cosmology — dynamical system approach

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Abstract

We investigate cosmological dynamics based on $f(R)$ gravity in the Palatini formulation. In this study we use the dynamical system methods. We show that the evolution of the Friedmann equation reduces to the form of the piece-wise smooth dynamical system. We demonstrate how the trajectories can be sewn to guarantee C^0 inextendibility of the metric. We point out that importance of dynamical system of Newtonian type with non-smooth right-hand sides in the context of Palatini cosmology. In this framework we can investigate naturally singularities which appear in the past and future of the cosmic evolution.

Outline

Dynamical systems

A dynamic model of a physical phenomenon can be described by a dynamical system which consists of

1. a space (state space or phase space),
2. a mathematical rule describing the evolution of any point in that space.

The state of the system is a set of quantities which are considered important about the system and the state space is the set of all possible values of these quantities.

The evolution of continuous dynamical systems is defined by ordinary differential equations (ODEs) in the form

$$\dot{x} = f(x)$$

where $x \in X$, i.e. x is element of state space $X \subset \mathbb{R}^n$, and function $f\colon X \to X$ is a vector field on $\mathbb{R}^n$ such that

$$f(x) = (f_1(x), \ldots, f_n(x))$$

and $x = (x_1, x_2, \ldots, x_n)$.

It is called an autonomous dynamical system as the vector field does not depend *explicite* on time.

For a qualitative approach to a dynamical system analysis it is necessary to define a critical point and the behaviour of solution near this point.

Definition

The autonomous equation $\dot{x} = f(x)$ is said to have a critical point or fixed point at $x = x_0$ if and only if $f(x_0) = 0$.

Definition

A critical point $x = x_0$ is stable (also called Lyapunov stable) if all solutions $x(t)$ starting near it stay close to it and asymptotically stable if it is stable and the solution approach the critical point for all nearby initial conditions.

Linear stability theory

Given a dynamical system $\dot{x} = f(x)$ with the critical point at $x = x_0$, the system is linearised about its critical point by

$$M = Df(x_0) = \left(\frac{\partial f_i}{\partial x_j} \right)_{x=x_0}$$

and the matrix M is called the Jacobi matrix.

The eigenvalues of this matrix for the system linearised about the critical point x_0 reveal stability/instability of that point provided that the point is hyperbolic.

Definition

Let $x = x_0$ be a critical point of the system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. Then x_0 is said to be hyperbolic if none of the eigenvalues of $Df(x_0)$ have zero real part, and non-hyperbolic otherwise.

The case of 2D system

Table: Hyperbolic critical points and their asymptotic stability

critical point	eigenvalues
stable node	$\lambda_1 < 0,\ \lambda_2 < 0$
unstable node	$\lambda_1 > 0,\ \lambda_2 > 0$
saddle	$\lambda_1,\ \lambda_2 \neq 0$, opposite sign
stable focus	$\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta$ with $\alpha < 0$ and $\beta \neq 0$
unstable focus	$\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta$ with $\alpha > 0$ and $\beta \neq 0$
centre	$\lambda_1 = i\beta,\ \lambda_2 = -i\beta$ with $\beta \neq 0$

If λ_1 or λ_2 or both are equal zero, the linearization technique does not allow to determine the stability of critical point.

Outline

Homogeneous cosmological models as dynamical systems

Einstein's field equations constitute, in general, a very complicated system of nonlinear, partial differential equations, but what is made use of in cosmology are the solutions with prior symmetry assumptions postulated at the very beginning.

In this case, the Einstein field equations can be reduced to a system of ordinary differential equation, i.e. a dynamical system. Hence, in cosmology the dynamical systems methods can be applied in a natural way. The applications of these methods allow to reveal some stability properties of particular solutions, visualized geometrically as trajectories in the phase space. Hence, one can see how large the class of the solutions leading to the desired property is, by means of attractors and the inset of limit set (an attractor is a limit set with an open inset—all the initial conditions that end up in some equilibrium state).

The dynamical system describes the behaviour of the Universe as a whole.

This dynamics should reproduce three periods of the Universe evolution

1. the early time: the expansion (inflation) from the unstable de Sitter state;
2. the present time: the matter domination epoch necessary for formation of the large-scale structure of the Universe;
3. the late time: the expansion of the Universe (asymptotically) to stable de Sitter state.

FRW models as dynamical system of Newtonian type

If we assume the validity of the Robertson-Walker symmetry for our Universe which is filled with perfect fluid satisfying the general form of the equation of state $p_{\rm eff} = w_{\rm eff}(a)\rho_{\rm eff}$, then $\rho_{\rm eff} = \rho_{\rm eff}(a)$, i.e. both the effective energy density $\rho_{\rm eff}$ and pressure $p_{\rm eff}$ are parameterized by the scale factor a as a consequence of the conservation condition

$$\dot{\rho}_{\rm eff} = -3H(\rho_{\rm eff} + p_{\rm eff}), \tag{1}$$

where dot denotes differentiation with respect to the cosmological time t and $H = d(\ln a)/dt$ is the Hubble function.

Let consider the acceleration (Raychaudhuri) equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_{\rm eff} + 3p_{\rm eff}). \tag{2}$$

We assume that the Universe is filled with standard dust matter (together with dark matter) and dark energy X

$$\begin{aligned} p_{\mathrm{eff}} &= 0 + w_X \rho_X, \\ \rho_{\mathrm{eff}} &= \rho_{\mathrm{m}} + \rho_X, \end{aligned} \tag{3}$$

where $w_X = w_X(a)$ is the coefficient of the equation of state for dark energy parameterized by the scale factor or redshift z: $1 + z = (\frac{a}{a_0})^{-1}$, where a_0 is the present value of the scale factor. For simplicity we assume $a_0 = 1$.

One can check that the Raychaudhuri equation (2) can be rewritten to the form analogous to the Newtonian equation

$$\ddot{a} = -\frac{\partial V}{\partial a}, \tag{4}$$

if we choose the following form of the potential function

$$V(a) = -\frac{1}{6}\rho_{\mathrm{eff}} a^2, \tag{5}$$

where ρ_{eff} satisfies the conservation condition (1).

As the alternative method to obtain (5) is integration by parts eq. (4) with the help of conservation condition (1).
The Raychaudhuri equation instead of the Friedmann first integral assumes the following form

$$\rho_{\mathrm{eff}} - 3\frac{\dot{a}^2}{a^2} = 3\frac{k}{a^2}, \tag{6}$$

or

$$\dot{a}^2 = -2V \tag{7}$$

where

$$V = -\frac{\rho_{\mathrm{eff}}\, a^2}{6} + \frac{k}{2}. \tag{8}$$

Equation (7) is the form of the first integral of the Einstein equation with Robertson-Walker symmetry called the Friedmann energy first integral. Formally, curvature effects can be incorporated into the effective energy density ($\rho_k = -\frac{k}{a^2}$).

The form of equation (4) suggests a possible interpretation of the evolutional paths of cosmological models as the motion of a fictitious particle of unit mass in a one dimensional potential parameterized by the scale factor. Following this interpretation the Universe is accelerating in the domain of configuration space $\{a\colon a \geq 0\}$ in which the potential is a decreasing function of the scale factor. In the opposite case, if the potential is an increasing function of a, the Universe is decelerating.

It is useful to represent the evolution of the system in terms of dimensionless density parameters $\Omega_i \equiv \rho_i/(3H_0^2)$, where H_0 is the present value of the Hubble function. For this aims it is sufficient to introduce the dimensionless scale factor $x \equiv a/a_0$ which measures the value of a in the units of the present value a_0, and parameterize the cosmological time following the rule $t \mapsto \tau\colon dt|H_0| = d\tau$. Note that this mapping is singular at $H_0 = 0$.

Hence we obtain a 2-dimensional dynamical system describing the evolution of cosmological models

$$\frac{dx}{d\tau} = y, \tag{9}$$

$$\frac{dy}{d\tau} = -\frac{\partial V}{\partial x}, \tag{10}$$

and $y^2/2 + V(x) = 0$, $1 + z = x^{-1}$. Where

$$V(x) = -\frac{1}{2}\left(\Omega_{\text{eff}} x^2 + \Omega_{k,0}\right),$$

$$\Omega_{\text{eff}} = \Omega_{\text{m},0} x^{-3} + \Omega_{X,0} x^{-3(1+w_X)},$$

for dust matter and quintessence matter satisfying the equation of state $p_X = w_X \rho_X$, $w_X = \text{const}$.
The system (9)-(10) opens the possibility of taking advantage of the dynamical systems methods to investigate all possible evolutional scenarios for all possible initial conditions.

Therefore, theoretical research in this area has obviously shifted from finding and analyzing particular cosmological solution to investigating a space of all admissible solutions and discovering how certain properties (like, for example, acceleration, existence of singularities) are “distributed” in this space.

The system (9)-(10) is a Hamiltonian one and adopting the Hamiltonian formalism to the admissible motion analysis seems to be natural. The analysis can then be performed in a manner similar to that of classical mechanics. The cosmology determines uniquely the form of the potential function $V(x)$, which is the central point of the investigations.

Different potential functions for different propositions of solving the acceleration problem are presented in Tables in two next slides.

Table: The potential functions for different dark energy models.

model	form of the potential function
Einstein-de Sitter model $\Omega_{\mathrm{m},0}=1$, $\Omega_{\Lambda,0}=0$, $\Omega_{k,0}=0$	$V(x)=-\frac{1}{2}\Omega_{\mathrm{m},0}x^{-1}$
ΛCDM model $\Omega_{\mathrm{m},0}+\Omega_{\Lambda,0}=1$	$V(x)=-\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1}+\Omega_{\Lambda,0}x^{2}\right)$
FRW model filled with n non-interacting multi-fluids $p=w\rho$ with dust matter	$V(x)=-\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1}+\Omega_{k,0}+\sum_{i=1}^{N}\Omega_{i,0}x^{-3(1+w_i)}\right)$
FRW quintessence model with dust and dark matter X $w_X<-1$ phantom models	$V(x)=-\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1}+\Omega_{k,0}+\Omega_{X,0}x^{-1-3w_X}\right)$
FRW model with generalized Chaplygin gas $p=-\frac{A}{\rho^{\alpha}}$, $A>0$	$V(x)=-\frac{1}{2}\left[\Omega_{\mathrm{m},0}x^{-1}+\Omega_{k,0}+\Omega_{\mathrm{Chapl},0}\left(A_S+\frac{1-A_S}{x^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}}\right]$
FRW models with dynamical equation of state for dark energy $p_X=w_X(a)\rho_X$ and dust	$V(x)=-\frac{1}{2}\left[\Omega_{\mathrm{m},0}x^{-1}+\Omega_{k,0}+\Omega_{X,0}x^{-1}\exp\left(\int_1^x\frac{w_X(a)}{a}da\right)\right]$
FRW models with dynamical equation of state for dark energy coefficient equation of state $w_X=w_0+w_1z$	$V(z)=-\frac{1}{2}\left[\Omega_{\mathrm{m},0}(1+z)+\Omega_{X,0}(1+z)^{1+3(w_0-w_1)}e^{3w_1z}+\Omega_{k,0}\right]$

Table: The potential functions for modified FRW equations.

model	form of the potential function
non-flat Cardassian models filled by dust matter	$V(x) = -\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1} + \Omega_{k,0} + \Omega_{\mathrm{Card},0}x^{m+2}\right)$
bouncing cosmological models $(H/H_0)^2 = \Omega_{\mathrm{m},0}x^{-m} - \Omega_{n,0}x^{-n}$, $n > m$	$V(x) = -\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-m+2} - \Omega_{n,0}x^{-n+2}\right)$
Randall-Sundrum brane models with dust on the brane and dark radiation	$V(x) = -\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1} + \Omega_{\lambda,0}x^{-6} + \Omega_{k,0} + \Omega_{d,0}x^{-4}\right)$
cosmology with spin and dust (MAG cosmology)	$V(x) = -\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1} + \Omega_{s,0}x^{-6} + \Omega_{k,0}\right)$
Dvali, Deffayet, Gabadadze brane models (DDG)	$V(x) = -\frac{1}{2}\left(\sqrt{\Omega_{\mathrm{m},0}x^{-1} + \Omega_{rc,0}} + \sqrt{\Omega_{rc,0}}\right)^2$
Sahni, Shtanov brane models	$V(x) = -\frac{1}{2}\Big(\Omega_{\mathrm{m},0}x^{-1} + \Omega_{\sigma,0} + 2\Omega_{l,0}$ $\pm 2\sqrt{\Omega_{l,0}}\sqrt{\Omega_{\mathrm{m},0}x^{-1} + \Omega_{\sigma,0} + \Omega_{l,0} + \Omega_{\Lambda_b,0}}\Big)$
FRW cosmological models of nonlinear gravity $\mathcal{L} \propto R^n$ with matter and radiation	$V(x) = -\frac{1}{2}\left[\frac{2n}{3-n}\Omega_{\mathrm{m},0}x^{-1} + \frac{4n(2-n)}{(n-3)^2}\Omega_{r,0}x^{-2}\right]\Omega_{\mathrm{nonl},0}x^{\frac{3(n-1)}{n}}$
ΛDGP model screened cosmological constant model	$V(x) = -\frac{1}{8}x^2\left[-\frac{1}{r_0H_0} + \sqrt{\left(2 + \frac{1}{r_0H_0}\right)^2 + 4\Omega_{\mathrm{m},0}(x^{-3} - 1)}\right]$

The key problem is to investigate the geometrical and topological properties of the ensemble of simple models of the universe which we define in the following way.

Definition

By the ensemble of simple models of the universe we understand the space of all 2-dimensional systems of the Newtonian type

$$\dot{x} = y,$$
$$\dot{y} = -\partial V/\partial x$$

with a suitably defined potential function of the scale factor, which characterizes the physical model of the dark energy or the modified FRW equation.

Because of the Newtonian form of the dynamical system the character of critical points is determined from the characteristic equation of the form

$$a^2 + \det A\Big|_{x_0=0, \frac{\partial V(a)}{\partial a}|_{a_0}=0} = 0, \tag{11}$$

where $\det A$ is determinant of linearization matrix calculated at the critical points, i.e.

$$\det A = \frac{\partial^2 V(a)}{\partial a^2}\Big|_{a_0, \frac{\partial V(a)}{\partial a}|_{a_0}=0}. \tag{12}$$

From equation (11) and (12) one can conclude that only admissible critical points are the saddle type if $\frac{\partial^2 V(a)}{\partial a^2}|_{a=a_0} < 0$ or centers type if $\frac{\partial^2 V(a)}{\partial a^2}|_{a=a_0} > 0$.

If a shape of the potential function is known (from the knowledge of effective energy density), then it is possible to calculate

- cosmological functions in exact form

$$t = \int^a \frac{da}{\sqrt{-2V(a)}}, \tag{13}$$

$$H(a) = \pm\sqrt{-\frac{2V(a)}{a^2}}, \tag{14}$$

- a deceleration parameter, an effective barotropic factor

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{1}{2}\frac{d\ln(-V)}{d\ln a}, \tag{15}$$

$$w_{\text{eff}}(a(t)) = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -\frac{1}{3}\left(\frac{d\ln(-V)}{d\ln a} + 1\right), \tag{16}$$

- a parameter of deviation from a de Sitter universe

$$h(t) \equiv -(q(t)+1) = \frac{1}{2}\frac{d\ln(-V)}{d\ln a} - 1 \tag{17}$$

(note that if $V(a) = -\frac{\Lambda a^2}{6}$, $h(t) = 0$),

- the effective matter density and pressure

$$\rho_{\text{eff}} = -\frac{6V(a)}{a^2}, \tag{18}$$

$$p_{\text{eff}} = \frac{2V(a)}{a^2}\left(\frac{d\ln(-V)}{d\ln a} + 1\right) \tag{19}$$

- and, finally, a Ricci scalar curvature for the FRW metric (**??**)

$$R = \frac{6V(a)}{a^2}\left(\frac{d\ln(-V)}{d\ln a} + 2\right). \tag{20}$$

From the formulas above one can observe that the most of them depend on the quantity

$$I_\nu(a) = \frac{d\ln(-V)}{d\ln a}. \tag{21}$$

This quantity measures elasticity of the potential function, i.e. indicates how the potential $V(a)$ changes if the scale factor a changes. For example, for the de Sitter universe $-V(a) \propto a^2$, the rate of growth of the potential is 2% as the rate of growth of the scale factor is 1%.
In the classification of the cosmological singularities by Fernandez-Jambrina and Lazkoz the crucial role is played by the parameter $h(t)$. Note that a cosmological sense of this parameter is

$$h(t) = \frac{1}{2} I_\nu(a) - 1. \tag{22}$$

In this approach the classification of singularities is based on generalized power and asymptotic expansion of the barotropic index w in the equation of state (or equivalently of the deceleration parameter q) in terms of the time coordinate.

Outline

1. Short introduction to dynamical systems
2. Dynamical systems approach to FRW cosmology
3. Lyapunov stability and Lyapunov function for FRW cosmological model
4. The phase portrait for dynamics of FRW cosmological model
5. Piece-wise smooth dynamical systems- examples from Palatini cosmology

Lyapunov stability

To prove the stability of the critical points of the system, the Lyapunov function can be used.

Definition

Given a smooth dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, and a critical point x_0, a continuous function $V\colon \mathbb{R}^n \to \mathbb{R}$ in a neighbourhood U of x_0 is a Lyapunov function for the point if

1. V is differentiable in $U \setminus \{x_0\}$,
2. $V(x) > V(x_0) \quad \forall x \in U \setminus \{x_0\}$,
3. $\dot{V} \leq 0 \quad \forall x \in U \setminus \{x_0\}$.

There is no formal way of constructing such a function. It can be done by an educated guess, by trial and error.

The existence of the Lyapunov function guarantees the asymptotic stability of the system.

Theorem

Let x_0 be a critical point of the system $\dot{x} = f(x)$, where $f : U \to \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ is a domain that contains x_0. If V is a Lyapunov function, then

1. *if $\dot{V} = \frac{\partial V}{\partial x} f$ is negative semi-definite, then $x = x_0$ is a stable fixed point,*
2. *if $\dot{V} = \frac{\partial V}{\partial x} f$ is negative definite, then $x = x_0$ is an asymptotically stable fixed point.*

Furthermore, if $||x|| \to \infty$ and $V(x) \to \infty$ for $\forall x$, then x_0 is said to be globally stable or globally asymptotically stable, respectively.

Lyapunov function for FRW cosmological model

We assume a cosmological model with topology $\mathbb{R} \times \mathcal{M}^3$ where $\mathcal{M}^3$ is maximally symmetric 3-space; the metric of spacetime is Lorenzian $(-,+,+,+)$ and assumes the following form

$$ds^2 = -dt^2 = a(t)^2 \left(\frac{dr^2}{1 - \frac{1}{4}kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \qquad (23)$$

where (t, r, θ, ϕ) are the pseudo-spherical coordinates (Wald, 1984), $a(t) > 0$ is the scale factor (the function of the time coordinate), and $k = -1, 0, 1$ is the spacial curvature. The 3-space of constant curvature is spatially open for $k = -1$, spatially closed for $k = 1$ and spatially flat for $k = 0$.

The dynamics of metric tensor $g_{\mu\nu}$: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ is determined from the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{24}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$, $(x^1, x^2, x^3, x^4) = (t, r, \theta, \phi)$, is the Ricci scalar; we use natural units such that $8\pi G = c = 1$.

Matter is assumed to be in the form of perfect fluid

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)u_\mu u_\nu \tag{25}$$

where $u^\mu = (-1, 0, 0, 0)$ denotes the four-velocity of an observer comoving with the fluid. The functions $p(t)$, $\rho(t)$ are pressure and energy density of the matter fluid, respectively.

Tensor energy-momentum can be generalised for non-perfect, viscous fluid (Weinberg, 1971), which for metric (23) assumes the form

$$T^0_{\ 0} = -\rho, \qquad T^i_{\ k} = \begin{cases} p - 3\xi\frac{\dot{a}}{a} & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \tag{26}$$

where $i, k = \{1, 2, 3\}$, ξ is the viscosity coefficient and $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter. Formally inclusion of viscous fluid is equivalent to replace pressure p by $p - 3\xi\frac{\dot{a}}{a}$ in the energy-momentum tensor.

Such a cosmological model with the imperfect fluid ($\xi = \text{const}$) has been considered since Heller et al. (1973) and Belinskii et al. (1975). The Einstein field equation (2) for this model reduces to (Szydlowski,1984)

$$\rho = -\Lambda + \frac{k}{a^2} + 3\frac{\dot{a}^2}{a^2} \tag{27}$$

$$p = \Lambda - 2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} \tag{28}$$

where Λ is the cosmological constant parameter. The dependence of pressure p on $H = \frac{\dot{a}}{a}$ means that we considered viscous effects with the viscosity coefficient $\xi = -\frac{1}{3}\frac{\partial p}{\partial H}$.
Equations (27)-(28) can be rewritten to the form of the three-dimensional autonomous dynamical system

$$\dot{H} = -H^2 - \frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3} \tag{29}$$

$$\dot{\rho} = -3H(\rho + p) \tag{30}$$

$$\dot{a} = Ha \tag{31}$$

where $p = p(H, \rho)$ is a general form of the assumed equation of state.

Let study the dynamical system (29)-(31) and the asymptotic stability of its solution ($H_0 = \sqrt{\frac{\Lambda}{3}}, \rho_0 = 0$) (future de Sitter solution).

Definition

A critical point $\mathbf{x}_0$ of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, is a (Lyapunov) stable point if for all neighbourhoods U of $\mathbf{x}_0$ there exists a neighbourhood U_* of $\mathbf{x}_0$ such that if $\mathbf{x}_0 \in U_*$ at $t = t_0$ then $\phi_t(\mathbf{x}_0) \in U$ for all $t > t_0$, where ϕ_t is the flow of a dynamical system. If the critical point $\mathbf{x}_0$ is stable for all $x \in U_*$, $\lim_{t\to\infty} ||\phi_t(\mathbf{x} - \mathbf{x}_0|| = 0$.

To determine the asymptotic stability of a solution of the dynamical system considered we construct the Lyapunov function (Perko, 2001).

Lyapunov function for FRW model

Let us consider some basic definitions of first integrals (Goriely, 2001).

Let $M \subset \mathbb{K}^n$ be an open subset in $\mathbb{K}^n$ where the field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. We denote by $\mathcal{X}(M)$ and $\mathcal{F}(M)$ the algebra of vector fields and functions on M, respectively. For simplicity, we assume that all objects are of the class C^∞. Let us consider a system of ordinary differential equations on M

$$\frac{dx}{dt} = X_F(x) = F(x), \quad x = (x^1, \ldots, x^n) \in M \subset \mathbb{K}^n \tag{32}$$

where the vector field $X_F = \mathcal{X}(M)$ is given by

$$X_F = \sum_{i=1}^{n} f^i(x) \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} f^i(x) \partial_i = f^i(x) \partial_i \tag{33}$$

where $(f^1(x), \ldots, f^n(x))$ are components of the vector field X_F.

We are looking for a solution or a class of solutions of system (32). This is the motivation of the following definition.

Definition

We say that subset $W \subset M$ is invariant with respect to system (32) if W consists only of the system's phase curves.

It seems that it is extremely difficult to check if a given set W is invariant with respect to (32) because, in general, we do not know its solutions. However, for checking the invariance, it is enough to know if, for all $x \in W$, the vector of phase velocity in this point is tangent to M, i.e. if $F(x) \in T_x W$ for all $x \in W$.

The most important invariant sets are those allowing to reduce the dimension of the system. For this purpose one invariant set is not enough, we need a one parameter family W_c of $(n-1)$-dimensional invariant sets that gives a foliation of M. Such a foliation arises naturally when we know a first integral.

Definition

Function $G \in \mathcal{F}(M)$ is called the first integral of system (32) if it is constant on all solutions of the system. It is equivalent to the condition

$$X_F(G)(x) = \partial_i G(x) f^i(x) = 0, \quad \text{for} \quad x \in M. \tag{34}$$

It is well known that a level $W_c = \{x \in M | G(x) = c\}$ of a first integral is invariant, and $c \to W_c$ gives the foliation mentioned earlier.

When we cannot find a first integral of the system, it is sometimes possible to find a function whose one level is invariant.

Definition (Lyapunov function)

Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in X \subset \mathbb{R}^n$ be a smooth autonomous system of equations with fixed point x_0. Let $V: \mathbb{R}^n \to \mathbb{R}$ be a continuous function in a neighbourhood U of $\mathbf{x}_0$, the V is called a Lyapunov function for the point x_0 if

1. V is differentiable in $U \setminus \{\mathbf{x}_0\}$
2. $V(\mathbf{x}) > V(\mathbf{x}_0)$
3. $\dot{V} \leq 0 \quad \forall \mathbf{x} \in U \setminus \{\mathbf{x}_0\}$.

Theorem (Lyapunov stability)

Let $\mathbf{x}_0$ be a critical point of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, and let U be a domain containing $\mathbf{x}_0$. If there exists a Lyapunov function $V(\mathbf{x})$ for which $\dot{V} \leq 0$, then $\mathbf{x}_0$ is a stable fixed point. If there exists a Lyapunov function $V(\mathbf{x})$ for which $\dot{V} < 0$, then $\mathbf{x}_0$ is a asymptotically stable fixed point. Furthermore, if $||x|| \to \infty$ and $V(\mathbf{x}) \to \infty$ for all $\mathbf{x}$, then $\mathbf{x}_0$ is said to be globally stable or globally asymptotically stable, respectively.

Let us return to our system (29)-(31).

Theorem

The system (29)-(31) has first integral in the form

$$\rho - 3H^2 + \Lambda = 3\frac{k}{a^2}. \tag{35}$$

Proof.

After differentiation of both sides of (10) with respect to time t and substitution of right-hand sides of (29)-(31) we obtain the form (35) of the first integral. □

In the three-dimensional phase space the first integral (10) defines surfaces for different values of the parameter Λ. The dimension of the dynamical system (29)-(31) can be lowered over one due to this first integral

$$\dot{H} = -H^2 - \frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3} \tag{36}$$

$$\dot{\rho} = -3H(\rho + p) \tag{37}$$

where $p = p(H, \rho)$, in generally.

For the de Sitter fixed point of (36)-(37), we have $p = -\rho$, from equation (37). Then from equation (36) and using the first integral (10) we obtain that $k = 0$. It means that fixed point is an intersection of the trajectory of the flat model and the line $\rho + p(H, \rho) = 0$ in the phase space (H, ρ).

Theorem

The de Sitter solution $H_0 = \pm\sqrt{\frac{\Lambda}{3}}$ is asymptotically stable for $H_0 > 0$ and asymptotically unstable for $H_0 < 0$.

Proof I

Let us propose the following Lyapunov function

$$V(H,\rho) \equiv \begin{cases} \rho - 3H^2 + \Lambda & \text{for } k = 0, 1 \\ -\left(\rho - 3H^2 + \Lambda\right) & \text{for } k = -1 \end{cases} \tag{38}$$

which can be obtained from (35) by putting $k = 0$. The surface $\{(H,\rho) \colon \rho - 3H^2 + \Lambda = 0\}$ divides the phase space into two domains occupied by the trajectories with $k = 1$ and $k = -1$, respectively.
Let us consider the first case of non-negative Lyapunov function $V(t)$ for $k = 0, 1$ in (38).

$$\begin{aligned} \dot{V}(t) &= \dot{\rho} - 6H\dot{H} = -3H(\rho + p) - 6H\left(-H^2 - \frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3}\right) \\ &= -3H\left\{\rho + p + 2\left[-H^2 - \frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3}\right]\right\} \\ &= -2H\left(\rho - 3H^2 + \Lambda\right) = -2H\frac{3k}{a^2} \leq 0, \quad \text{if } H > 0. \end{aligned} \tag{39}$$

Proof II

Analogously, we choose the second case of Lyapunov function $V(t)$ for $k = -1$ in (38) to have the function $V(t)$ to be non-negative.
Finally, we obtain that at both critical points $(H = \pm\sqrt{\frac{\Lambda}{3}}, \rho_0 = 0)$ the Lyapunov function (38) vanishes.
So, the conditions of the Lyapunov stability theorem are satisfied.

We conclude that while the stable de Sitter solution is asymptotically stable, the unstable de Sitter solution is unstable. This result was obtained by using global methods of dynamics investigations instead of the standard local stability analysis.

The choice of the Lyapunov function in the form of a first integral is suitable for proving asymptotic stability of the stable de Sitter solution of the model. This methodological result has also clear cosmological interpretation: the stable de Sitter universe has no hair like a black hole.

Outline

Dynamics of FRW cosmological model

Let us consider a homogeneous and isotropic Friedmann-Robertson-Walker model with metric (23) and matter with energy density ρ and the cosmological constant Λ. We choose two phase space variables: the Hubble parameter $H = x$ and energy density $\rho = y$ and define the dynamical system

$$\dot{x} = -x^2 - \frac{1}{6}y + \frac{\Lambda}{3} \tag{40}$$

$$\dot{y} = -3xy \tag{41}$$

where the dot denotes derivative with respect to time t and $\Lambda > 0$ is the cosmological constant. It is a special case of system (36)-(37).

Let us apply the local stability analysis for system (40)-(41).

Remark

System (40)-(41) has three critical points: stable node $(x = -\sqrt{\frac{\Lambda}{3}}, y = 0)$*, unstable node* $(x = \sqrt{\frac{\Lambda}{3}}, y = 0)$*, and a saddle* $(x = 0, y = 2\Lambda)$*.*

From the characteristic equation $\det(A - \lambda I) = 0$, where the linearization matrix of system (40)-(41) is

$$A = \begin{bmatrix} -2x_0 & \frac{1}{6} \\ -3y_0 & -3x_0 \end{bmatrix} \qquad (42)$$

we have that determinant, trace of linearization matrix A and discriminant of the characteristic equation are $\det A = 6x_0^2 - \frac{1}{2}y_0$, $\operatorname{tr} A = -5x_0$ and $\Delta = x_0^2 + 2y_0$, respectively, where (x_0, y_0) is a critical point.

Therefore,
1. the critical point $(x_0 = -\sqrt{\frac{\Lambda}{3}}, y_0 = 0)$ is an unstable node as $\det A = 2\Lambda > 0$, $\operatorname{tr} A = \frac{5}{3}\sqrt{\Lambda} > 0$, and $\Delta = \frac{\Lambda}{3} > 0$;
2. the critical point $(x_0 = \sqrt{\frac{\Lambda}{3}}, y_0 = 0)$ is a stable node as $\det A = -\Lambda < 0$, $\operatorname{tr} A = -\frac{5}{3}\sqrt{\Lambda} < 0$, and $\Delta = \frac{\Lambda}{3} > 0$;
3. the critical point $(x_0 = 0, y_0 = 2\Lambda)$ is a saddle as $\det A = 2\Lambda > 0$, $\operatorname{tr} A = 0$, and $\Delta = 4\Lambda > 0$.

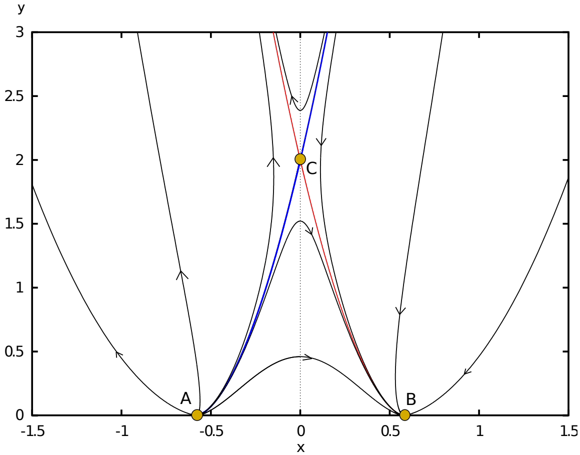

Figure: The phase portrait of system (94)-(95). There three critical points: point A represents the unstable de Sitter universe, point B represents the stable de Sitter universe, and point C represents the Einstein-de Sitter universe. The red and blue trajectories lie on unstable and stable invariant manifolds, respectively. It is assumed Λ is positive (for illustration $\Lambda = 1$).

Outline

The Palatini approach

In the Palatini gravity action for $f(\hat{R})$ gravity is introduced to be

$$S = S_{\mathrm{g}} + S_{\mathrm{m}} = \frac{1}{2}\int \sqrt{-g} f(\hat{R}) d^4x + S_{\mathrm{m}}, \tag{43}$$

where $\hat{R} = g^{\mu\nu}\hat{R}_{\mu\nu}(\Gamma)$ is the generalized Ricci scalar and $\hat{R}_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsionless connection Γ. Hereafter, we assume that $8\pi G = c = 1$. The equation of motion obtained from the first order Palatini formalism reduces to

$$f'(\hat{R})\hat{R}_{\mu\nu} - \frac{1}{2} f(\hat{R}) g_{\mu\nu} = T_{\mu\nu}, \tag{44}$$

$$\hat{\nabla}_{\alpha}(\sqrt{-g} f'(\hat{R}) g^{\mu\nu}) = 0, \tag{45}$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta L_{\mathrm{m}}}{\delta g_{\mu\nu}}$ is matter energy momentum tensor, i.e. one assumes that the matter minimally couples to the metric.

As a consequence the energy momentum tensor is conserved, i.e.: $\nabla^\mu T_{\mu\nu} = 0$. In eq. (45) $\hat{\nabla}_\alpha$ means the covariant derivative calculated with respect to Γ. In order to solve equation (45) it is convenient to introduce a new metric

$$\sqrt{h}h_{\mu\nu} = \sqrt{-g}f'(\hat{R})g_{\mu\nu} \tag{46}$$

for which the connection $\Gamma = \Gamma_{LC}(h)$ is a Levi-Civita connection. As a consequence in dim $M = 4$ one gets

$$h_{\mu\nu} = f'(\hat{R})g_{\mu\nu}, \tag{47}$$

i.e. that both metrics are related by the conformal factor. For this reason one should assume that the conformal factor $f'(\hat{R}) \neq 0$, so it has strictly positive or negative values.

Taking the trace of (44), we obtain additional so called structural equation

$$f'(\hat{R})\hat{R} - 2f(\hat{R}) = T. \tag{48}$$

where $T = g^{\mu\nu}T_{\mu\nu}$. Because of cosmological applications we assume that the metric g is the FRW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{1}{1-kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{49}$$

where $a(t)$ is the scale factor, k is a constant of spatial curvature ($k = 0, \pm 1$), t is the cosmological time. For simplicity of presentation we consider the flat model ($k = 0$).

As a source of gravity we assume perfect fluid with the energy-momentum tensor

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho, p, p, p), \tag{50}$$

where $p = w\rho$, $w = const$ is a form of the equation of state ($w = 0$ for dust and $w = 1/3$ for radiation). Formally, effects of the spatial curvature can be also included to the model by introducing a curvature fluid $\rho_{\mathrm{k}} = -\frac{k}{2}a^{-2}$, with the barotropic factor $w = -\frac{1}{3}$ ($p_{\mathrm{k}} = -\frac{1}{3}\rho_{\mathrm{k}}$). From the conservation condition $T^{\mu}_{\nu;\mu} = 0$ we obtain that $\rho = \rho_0 a^{-3(1+w)}$. Therefore trace T reads as

$$T = \sum_i \rho_{i,0}(3w_i - 1)a(t)^{-3(1+w_i)}. \tag{51}$$

In what follows we consider visible and dark matter ρ_{m} in the form of dust $w = 0$, dark energy ρ_{Λ} with $w = -1$ and radiation ρ_{r} with $w = 1/3$.

Because a form of the function $f(\hat{R})$ is unknown, one needs to probe it via ensuing cosmological models. Here we choose the simplest modification of the general relativity Lagrangian

$$f(\hat{R}) = \hat{R} + \gamma \hat{R}^2, \tag{52}$$

induced by first three terms in the power series decomposition of an arbitrary function $f(R)$. In fact, since the terms $\hat{R}^n$ have different physical dimensions, i.e. $[\hat{R}^n] \neq [\hat{R}^m]$ for $n \neq m$, one should take instead the function $\hat{R}_0 f(\hat{R}/\hat{R}_0)$ for constructing our Lagrangian, where $\hat{R}_0$ is a constant and $[\hat{R}_0] = [\hat{R}]$. In this case the power series expansion reads: $\hat{R}_0 f(\hat{R}/\hat{R}_0) = \hat{R}_0 \sum_{n=0} \alpha_n (\hat{R}/\hat{R}_0)^n = \sum_{n=0} \tilde{\alpha}_n \hat{R}^n$, where the coefficients α_n are dimensionless, while $[\tilde{\alpha}_n] = [\hat{R}]^{1-n}$ are dimension full.

On the other hand the Lagrangian (52) can be viewed as a simplest deviation, by the quadratic Starobinsky term, from the Lagrangian $\hat{R}$ which provides the standard cosmological model, a.k.a. the ΛCDM model. A corresponding solution of the structural equation (48)

$$\hat{R} = -T \equiv 4\rho_{\Lambda,0} + \rho_{\mathrm{m},0} a^{-3} \tag{53}$$

is, in fact, exactly the same as for the ΛCDM model, i.e. with $\gamma = 0$. However, the Friedmann equation of the ΛCDM model (with dust matter, dark energy and radiation)

$$H^2 = \frac{1}{3}\left(\rho_{\mathrm{r},0} a^{-4} + \rho_{\mathrm{m},0} a^{-3} + \rho_{\Lambda,0}\right) \tag{54}$$

is now hardly affected by the presence of the quadratic term.

More exactly a counterpart of the above formula in the model under consideration looks as follows

$$\frac{H^2}{H_0^2} = \frac{b^2}{\left(b+\frac{d}{2}\right)^2}\left[\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3}+\Omega_{\Lambda,0})^2\frac{(K-3)(K+1)}{2b}\right.$$
$$\left. +(\Omega_{\mathrm{m},0}a^{-3}+\Omega_{\Lambda,0})+\frac{\Omega_{\mathrm{r},0}a^{-4}}{b}+\Omega_k\right], \quad (55)$$

where

$$\Omega_k = -\frac{k}{H_0^2a^2}, \qquad \Omega_{\mathrm{r},0} = \frac{\rho_{\mathrm{r},0}}{3H_0^2}, \qquad \Omega_{\mathrm{m},0} = \frac{\rho_{\mathrm{m},0}}{3H_0^2}, \quad (56)$$

$$\Omega_{\Lambda,0} = \frac{\rho_{\Lambda,0}}{3H_0^2}, \qquad K = \frac{3\Omega_{\Lambda,0}}{(\Omega_{\mathrm{m},0}a^{-3}+\Omega_{\Lambda,0})}, \qquad \Omega_\gamma = 3\gamma H_0^2, \quad (57)$$

$$b = f'(\hat{R}) = 1+2\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3}+4\Omega_{\Lambda,0}), \quad (58)$$

$$d = \frac{1}{H}\frac{db}{dt} = -2\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3}+\Omega_{\Lambda,0})(3-K) \quad (59)$$

From the above one can check that the standard model (54) can be reconstructed in the limit $\gamma \mapsto 0$.

Degenerated singularities – new type (VI) of singularity – sewn singularities

Recently, due to discovery of an accelerated phase in the expansion of our Universe, many theoretical possibilities for future singularity are seriously considered. If we assume that the Universe expands following the Friedmann equation, then this phase of expansion is driven by dark energy – hypothetical fluid, which violates the strong energy condition. Many of new types of singularities were classified by Nojiri et al.. Following their classification the type of singularity depends on the singular behavior of the cosmological quantities like: the scale factor a, the Hubble parameter H, the pressure p and the energy density ρ.

- Type 0: 'Big crunch'. In this type, the scale factor a is vanishing and blow up of the Hubble parameter H, energy density ρ and pressure p.
- Type I: 'Big rip'. In this type, the scale factor a, energy density ρ and pressure p are blown up.
- Type II: 'Sudden'. The scale factor a, energy density ρ and Hubble parameter H are finite and $\dot{H}$ and the pressure p are divergent.
- Type III: 'Big freeze'. The scale factor a is finite and the Hubble parameter H, energy density ρ and pressure p are blown up or divergent.
- Type IV. The scale factor a, Hubble parameter H, energy density ρ, pressure p and $\dot{H}$ are finite but higher derivatives of the scale factor a diverge.
- Type V. The scale factor a is finite but the energy density ρ and pressure p vanish.

Following Królak, big rip and big crunch singularities are strong whereas sudden, big freeze and type IV are weak singularities.
In the model under consideration the potential function or/and its derivative can diverge at isolated points (value of the scale factor). Therefore mentioned before classification has application only for a single component of piece-wise smooth potential. In our model the dynamical system describing evolution of a universe belongs to the class of a piecewise smooth dynamical systems. As a consequence new types of singularities at finite scale factor a_s can appear for which $\frac{\partial V}{\partial a}(a_s)$ does not exist (is not determined). This implies that the classification of singularities should be extended to the case of non-isolated singularities.

Let us illustrate this idea on the example of freeze singularity in the Starobinsky model with the Palatini formalism (previous section). Such a singularity has a complex character and in analogy to the critical point we called it degenerated. Formally it is composed of two types III singularities: one in the future and another one in the past. If we considered the evolution of the universe before this singularity we detect isolated singularity of type III in the future. Conversely if we consider the evolution after the singularity then going back in time we meet type III singularity in the past. Finally, at the finite scale factor both singularities will meet together. For description of behavior near the singularity one considers $t = t(a)$ relation. This relation has a horizontal inflection point and it is natural to expand this relation in the Taylor series near this point at which $\frac{dt}{da} = \frac{1}{Ha}$ is zero. For the freeze singularity, the scale factor remains constant a_s, ρ and H blow up and $\ddot{a}$ is undefined. It this case, the degenerated singularity of type III is called sewn (non-isolated) singularity.

We, therefore, obtain

$$t - t_s \simeq \pm \frac{1}{2}\frac{d^2 t}{da^2}\Big|_{a=a_{\rm sing}} (a - a_{\rm sing})^2. \qquad (60)$$

Above formula combine two types of behavior near the freeze singularities in the future

$$a - a_{\rm sing} \propto -(t_{\rm sing} - t)^{1/2} \text{ for } t \to t_{\rm sing^-} \qquad (61)$$

and in the past

$$a - a_{\rm sing} \propto +(t - t_{\rm sing})^{1/2} \text{ for } t \to t_{\rm sing^+}. \qquad (62)$$

In the model under consideration another type of sewn singularity also appears. It is a composite singularity with two sudden singularities glued at the bounce when $a = a_{\rm min}$. In this singularity the potential itself is a continuous function while its first derivative has a discontinuity. Therefore, the corresponding dynamical system represents a piece-wise smooth dynamical system.

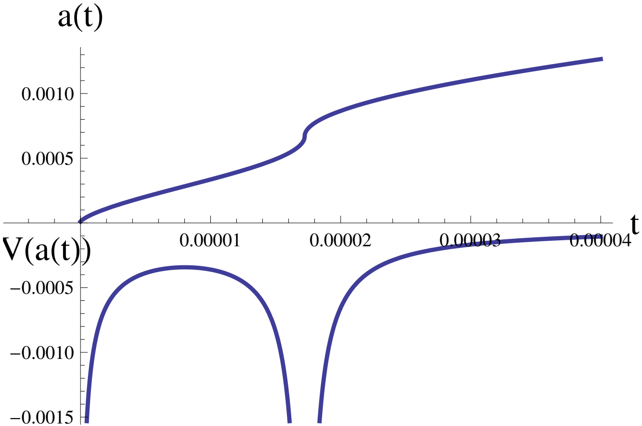

Figure: Illustration of sewn freeze singularity, when the potential $V(a)$ has a pole.

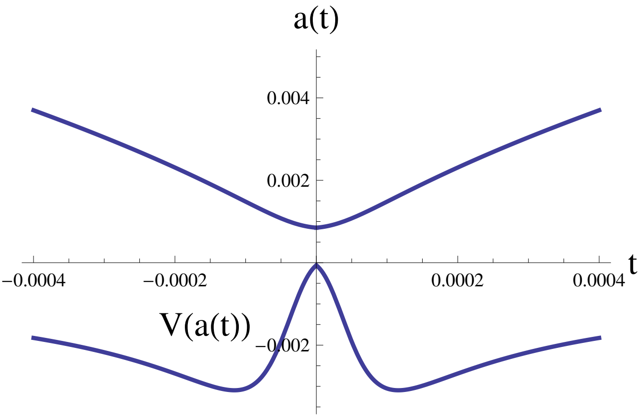

Figure: Illustration of a sewn sudden singularity. The model with negative Ω_γ has a mirror symmetry with respect to the cosmological time. Note that the spike on the diagram shows discontinuity of the function $\frac{\partial V}{\partial a}$. Note the existence of a bounce at $t = 0$.

Singularities in the Starobinsky model in the Palatini formalism

In our model, one finds two types of singularities, which are a consequence of the Palatini formalism: the freeze and sudden singularity. The freeze singularity appears when the multiplicative expression $\frac{b}{b+d/2}$, in the Friedmann equation (55), is equal the infinity. So we get a condition for the freeze singularity: $2b+d=0$ which produces a pole in the potential function. It appears that the sudden singularity appears in our model when the multiplicative expression $\frac{b}{b+d/2}$ vanishes. This condition is equivalent to the case $b=0$.

The freeze singularity in our model is a solution of the algebraic equation

$$2b+d=0 \Longrightarrow f(K,\Omega_{\Lambda,0},\Omega_\gamma)=0 \tag{63}$$

or

$$-3K-\frac{K}{3\Omega_\gamma(\Omega_\mathrm{m}+\Omega_{\Lambda,0})\Omega_{\Lambda,0}}+1=0, \tag{64}$$

where $K\in[0,\ 3)$.

The solution of the above equation is

$$K_{\text{freeze}} = \frac{1}{3 + \frac{1}{3\Omega_\gamma(\Omega_{\text{m}}+\Omega_{\Lambda,0})\Omega_{\Lambda,0}}}. \tag{65}$$

From equation (65), we can find an expression for a value of the scale factor for the freeze singularity

$$a_{\text{freeze}} = \left(\frac{1-\Omega_{\Lambda,0}}{8\Omega_{\Lambda,0} + \frac{1}{\Omega_\gamma(\Omega_{\text{m}}+\Omega_{\Lambda,0})}}\right)^{\frac{1}{3}}. \tag{66}$$

The relation between a_{freeze} and positive Ω_γ is presented in Figure 4. The sudden singularity appears when $b = 0$. This provides the following algebraic equation

$$1 + 2\Omega_\gamma(\Omega_{\text{m},0}a^{-3} + \Omega_{\Lambda,0})(K+1) = 0. \tag{67}$$

The above equation can be rewritten as

$$1 + 2\Omega_\gamma(\Omega_{\text{m},0}a^{-3} + 4\Omega_{\Lambda,0}) = 0. \tag{68}$$

From equation (68), we have the formula for the scale factor for a sudden singularity

$$a_{\mathrm{sudden}} = \left(-\frac{2\Omega_{\mathrm{m},0}}{\frac{1}{\Omega_\gamma} + 8\Omega_{\Lambda,0}} \right)^{1/3}. \tag{69}$$

which, in fact, becomes a (degenerate) critical point and a bounce at the same time.

Let
$V = -\frac{a^2}{2}\left(\Omega_\gamma(\Omega_{\mathrm{m},0}a^{-3} + 4\Omega_{\Lambda,0})^{2\frac{(K-3)(K+1)}{2b}} + (\Omega_{\mathrm{m},0}a^{-3} + 4\Omega_{\Lambda,0}) + \Omega_k\right)$.
We can rewrite dynamical system (9)-(10) as

$$a' = y, \tag{70}$$

$$y' = -\frac{\partial V(a)}{\partial a}, \tag{71}$$

where $' \equiv \frac{d}{d\sigma} = \frac{b+\frac{d}{2}}{b}\frac{d}{d\tau}$ is a new parametrization of time.

We can treated the dynamical system (70)-(71) as a sewn dynamical system. In this case, we divide the phase portrait into two parts: the first part is for $a < a_{\mathrm{sing}}$ and the second part is for $a > a_{\mathrm{sing}}$. Both parts are glued along the singularity.
For $a < a_{\mathrm{sing}}$, dynamical system (70)-(71) can be rewritten to the corresponding form

$$a' = y, \tag{72}$$
$$y' = -\frac{\partial V_1(a)}{\partial a}, \tag{73}$$

where $V_1 = V(-\eta(a - a_s) + 1)$ and $\eta(a)$ notes the Heaviside function.
For $a > a_{\mathrm{sing}}$, in the analogous way, we get the following equations

$$a' = y, \tag{74}$$
$$y' = -\frac{\partial V_2(a)}{\partial a}, \tag{75}$$

where $V_2 = V\eta(a - a_s)$.

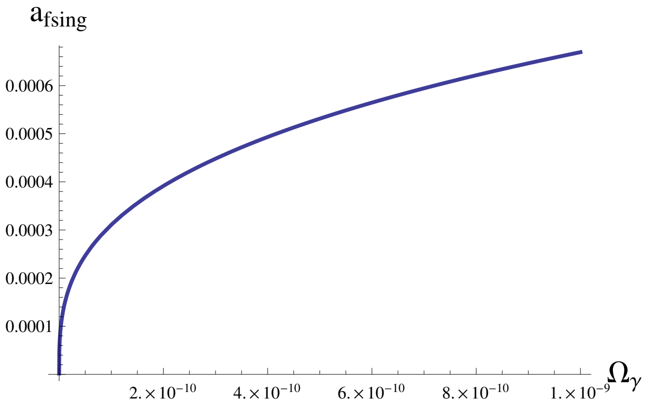

Figure: Diagram of the relation between a_{sing} and positive Ω_γ. Note that in the limit $\Omega_\gamma \mapsto 0$ the singularity overlaps with a big-bang singularity.

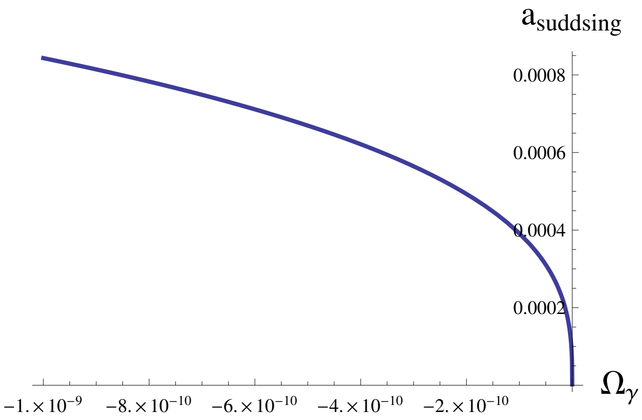

Figure: Diagram of the relation between a_{sing} and negative Ω_γ. Note that in the limit $\Omega_\gamma \mapsto 0$ the singularity overlaps with a big-bang singularity.

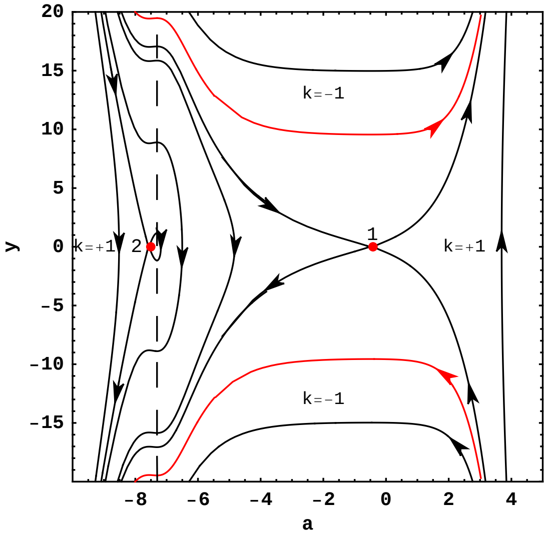

Figure: The phase portrait of the system (70-71) for positive Ω_γ. The scale factor a is in the logarithmic scale.

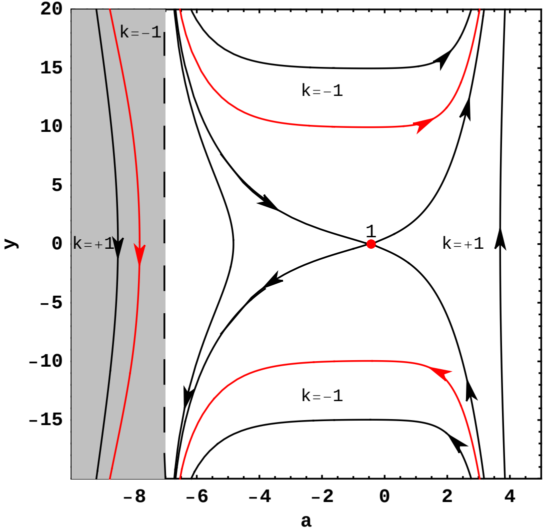

Figure: The phase portrait of the system (70-71) for negative Ω_γ. The scale factor a is in logarithmic scale.

The Palatini approach in the Einstein frame

The action (43) is dynamically equivalent to the first order Palatini gravitational action, provided that $f^{''}(\hat{R}) \neq 0$

$$S(g_{\mu\nu}, \Gamma^{\lambda}_{\rho\sigma}, \chi) = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left(f'(\chi)(\hat{R} - \chi) + f(\chi) \right) + S_m(g_{\mu\nu}, \psi), \quad (76)$$

Introducing a scalar field $\Phi = f'(\chi)$ and taking into account the constraint $\chi = \hat{R}$, one gets the action (76) in the following form

$$S(g_{\mu\nu}, \Gamma^{\lambda}_{\rho\sigma}, \Phi) = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left(\Phi \hat{R} - U(\Phi) \right) + S_m(g_{\mu\nu}, \psi), \quad (77)$$

where the potential $U(\Phi)$ is defined by

$$U_f(\Phi) \equiv U(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi)) \quad (78)$$

with $\Phi = \frac{df(\chi)}{d\chi}$ and $\hat{R} \equiv \chi = \frac{dU(\Phi)}{d\Phi}$.

The Palatini variation of the action (77) gives rise to the following equations of motion

$$\Phi\left(\hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R}\right) + \frac{1}{2}g_{\mu\nu}U(\Phi) - T_{\mu\nu} = 0, \tag{79a}$$

$$\hat{\nabla}_\lambda(\sqrt{-g}\Phi g^{\mu\nu}) = 0, \tag{79b}$$

$$\hat{R} - U'(\Phi) = 0. \tag{79c}$$

Equation (79b) implies that the connection $\hat{\Gamma}$ is a metric connection for a new metric $\bar{g}_{\mu\nu} = \Phi g_{\mu\nu}$; thus $\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}, \bar{R} = \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} = \Phi^{-1}\hat{R}$ and $\bar{g}_{\mu\nu}\bar{R} = g_{\mu\nu}\hat{R}$. The g-trace of (79a) produces a new structural equation

$$2U(\Phi) - U'(\Phi)\Phi = T. \tag{80}$$

Now equations (79a) and (79c) take the following form

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{U}(\Phi), \tag{81}$$

$$\Phi\bar{R} - (\Phi^2\,\bar{U}(\Phi))' = 0, \tag{82}$$

where we introduce $\bar{U}(\phi) = U(\phi)/\Phi^2$, $\bar{T}_{\mu\nu} = \Phi^{-1}T_{\mu\nu}$ and the structural equation can be replaced by

$$\Phi\,\bar{U}'(\Phi) + \bar{T} = 0\,. \tag{83}$$

In this case, the action for the metric $\bar{g}_{\mu\nu}$ and scalar field Φ is given by

$$S(\bar{g}_{\mu\nu},\Phi) = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-\bar{g}}\left(\bar{R} - \bar{U}(\Phi)\right) + S_m(\Phi^{-1}\bar{g}_{\mu\nu},\psi), \tag{84}$$

where we have to take into account a non-minimal coupling between Φ and $\bar{g}_{\mu\nu}$

$$\bar{T}^{\mu\nu} = -\frac{2}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}_{\mu\nu}}S_m = (\bar{\rho}+\bar{p})\bar{u}^{\mu}\bar{u}^{\nu} + \bar{p}\bar{g}^{\mu\nu} = \Phi^{-3}T^{\mu\nu}\ , \tag{85}$$

$\bar{u}^{\mu} = \Phi^{-\frac{1}{2}}u^{\mu}$, $\bar{\rho} = \Phi^{-2}\rho$, $\bar{p} = \Phi^{-2}p$, $\bar{T}_{\mu\nu} = \Phi^{-1}T_{\mu\nu}$, $\bar{T} = \Phi^{-2}T$.

In the FRW case, one gets the metric $\bar{g}_{\mu\nu}$ in the following form

$$d\bar{s}^2 = -d\bar{t}^2 + \bar{a}^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{86}$$

where $d\bar{t} = \Phi(t)^{\frac{1}{2}}dt$ and new scale factor $\bar{a}(\bar{t}) = \Phi(\bar{t})^{\frac{1}{2}}a(\bar{t})$. Ensuing cosmological equations (in the case of the barotropic matter) are given by

$$3\bar{H}^2 = \bar{\rho}_\Phi + \bar{\rho}_m, \qquad 6\frac{\ddot{\bar{a}}}{\bar{a}} = 2\bar{\rho}_\Phi - \bar{\rho}_m(1+3w) \tag{87}$$

where

$$\bar{\rho}_\Phi = \frac{1}{2}\bar{U}(\Phi), \qquad \bar{\rho}_\mathrm{m} = \rho_0\bar{a}^{-3(1+w)}\Phi^{\frac{1}{2}(3w-1)} \tag{88}$$

and $w = \bar{p}_\mathrm{m}/\bar{\rho}_\mathrm{m} = p_\mathrm{m}/\rho_\mathrm{m}$. In this case, the conservation equations has the following form

$$\dot{\bar{\rho}}_\mathrm{m} + 3\bar{H}\bar{\rho}_\mathrm{m}(1+w) = -\dot{\bar{\rho}}_\Phi. \tag{89}$$

Let us consider the Starobinsky–Palatini model in the above framework. The potential $\bar{U}$ is described by the following formula

$$\bar{U}(\Phi) = 2\bar{\rho}_\Phi(\Phi) = \left(\frac{1}{4\gamma} + 2\lambda\right)\frac{1}{\Phi^2} - \frac{1}{2\gamma}\frac{1}{\Phi} + \frac{1}{4\gamma}. \tag{90}$$

Cosmological equation for the Starobinsky–Palatini model in the Einstein frame can be rewritten to the form of the dynamical system in the variables $\bar{H}(\bar{t})$ and $\hat{R}(\bar{t})$

$$\dot{\bar{H}}(\bar{t}) = \frac{1}{6\ (1+2\gamma\hat{R}(\bar{t}))^2}$$
$$\Big(6\Lambda - 6\bar{H}(\bar{t})^2(1+2\gamma\hat{R}(\bar{t}))^2 + \hat{R}(\bar{t})(-1+24\gamma\Lambda+\gamma(1+24\gamma\Lambda)\hat{R}(\bar{t}))\Big)\,, \tag{91}$$

$$\dot{\hat{R}}(\bar{t}) = -\frac{3}{(-1+\gamma\hat{R}(\bar{t}))}\ \bar{H}(\bar{t})(1+2\gamma\hat{R}(\bar{t}))$$
$$\Big(4\Lambda + \hat{R}(\bar{t})\left(-1+16\gamma\Lambda+16\gamma^2\Lambda\hat{R}(\bar{t})\right)\Big)\,, \tag{92}$$

where a dot means the differentiation with respect to the time $\bar{t}$.

Equations (91)–(92) have the following the first integral in the following form:

$$\bar{H}(\bar{t})^2 + \Lambda - \frac{\hat{R}(\bar{t})(2+\gamma\hat{R}(\bar{t}))}{6(1+2\gamma\hat{R}(\bar{t}))^2} + k\frac{e^{-\frac{\arctan\left(\frac{-1+16\gamma\Lambda+32\gamma^2\Lambda\hat{R}(\bar{t})}{\sqrt{-1+32\gamma\Lambda}}\right)}{3\sqrt{-1+32\gamma\Lambda}}}\sqrt{4\Lambda+\hat{R}(\bar{t})\left(-1+16\gamma\Lambda+16\gamma^2\Lambda\hat{R}(\bar{t})\right)}}{C_0(1+2\gamma\hat{R}(\bar{t}))} = 0, \quad (93)$$

where $C_0 = \frac{a_0^2 e^{-\frac{\arctan\left(\frac{-1+16\gamma\Lambda+32\gamma^2\Lambda\hat{R}(\bar{t}_0)}{\sqrt{-1+32\gamma\Lambda}}\right)}{3\sqrt{-1+32\gamma\Lambda}}}\sqrt{4\Lambda+\hat{R}(\bar{t}_0)\left(-1+16\gamma\Lambda+16\gamma^2\Lambda\hat{R}(\bar{t}_0)\right)}}{(1+2\gamma\hat{R}(\bar{t}_0))}$.
Here, a_0 is the present value of the scale factor.

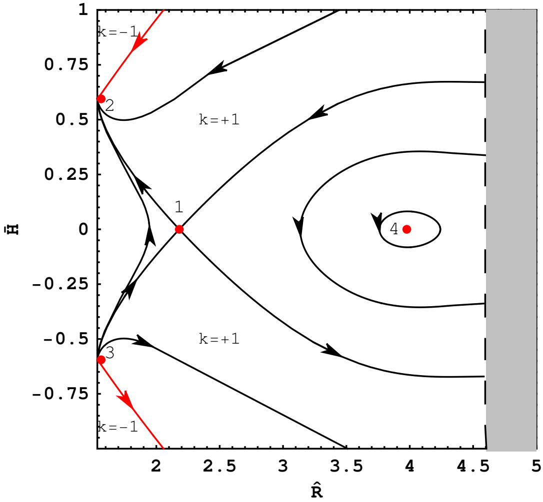

Figure: The phase portrait of system (91)-(92).

For comparison of the dynamical system in the both frames, we obtain dynamical system for the Starobinsky–Palatini model in the variables $H(t)$ and $\hat{R}(t)$

$$\dot{H}(t) = -\frac{1}{6}\left[6\left(2\Lambda + H(t)^2\right) + \hat{R}(t) + \frac{18(1+8\gamma\Lambda)\left(\Lambda - H(t)^2\right)}{-1-12\gamma\Lambda+\gamma\hat{R}(t)} - \frac{18(1+8\gamma\Lambda)H(t)^2}{1+2\gamma\hat{R}(t)}\right], \quad (94)$$

$$\dot{\hat{R}}(t) = -3H(t)(\hat{R}(t) - 4\Lambda), \quad (95)$$

where a dot means the differentiation with respect to time t.

Equations (94)–(95) have the following the first integral in the following form:

$$H(t)^2 - \frac{(1+2\gamma\hat{R}(t))^2\left(-3\Lambda + \hat{R}(t) - \frac{k(-4\Lambda+\hat{R}(t))^{2/3}}{C_0} + \frac{\gamma(12\Lambda-3\hat{R}(t))\hat{R}(t)}{2(1+2\gamma\hat{R}(t))}\right)}{(1+2\gamma\hat{R}(t)-3\gamma(-4\Lambda+\hat{R}(t)))^2} = 0, \quad (96)$$

where $C_0 = a_0^2(-4\Lambda + \hat{R}(t_0))^{2/3}$. Here, a_0 is the present value of the scale factor.

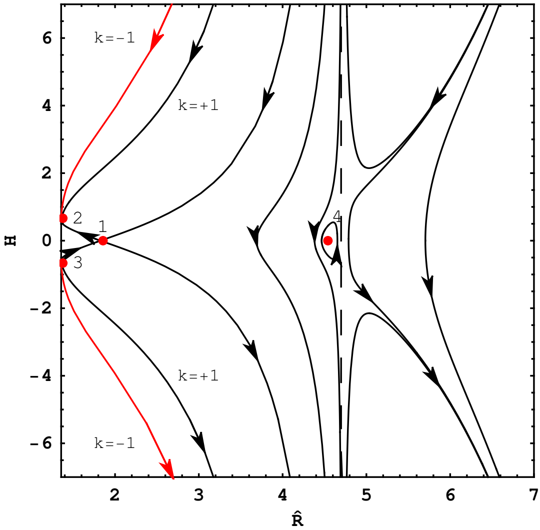

Figure: The phase portrait of system (94)-(95).

Summary: the cosmology as science about the Universe I

1. Methods of dynamical systems are very useful in cosmology as they allow to investigate all solutions for all possible initial conditions. The Universe is an unique entity with unknown initial conditions.
2. In the geometric language of phase space the “distribution” of models in the ensemble can be described. Are they generic (typical in any sense) or exceptional (fine tuned)? Problem is mathematically interesting as it requires a definition of probabilistic measure in the ensemble. Initial conditions for the Universe were special or typical?
3. The stability was studied by using global methods of dynamics investigations instead of the standard local stability analysis. The choice of the Lyapunov function in the form of a first integral is suitable for proving asymptotic stability of the stable de Sitter solution of the model. This methodological result has also clear cosmological interpretation: the stable de Sitter universe has no hair like a black hole.

Summary: the cosmology as science about the Universe II

4. In the context of the Starobinsky model in the Palatini formalism we found a new type of double singularity beyond the well-known classification of isolated singularities.
5. The phase portrait for the Starobinsky model in the Palatini formalism with a positive value of γ is equivalent to the phase portrait of the ΛCDM model. There is only a quantitative difference related with the presence of the non-isolated freeze singularity.
6. For the Starobinsky–Palatini model in the Einstein frame for the positive γ parameter, the sewn freeze singularity are replaced by the generalized sudden singularity. In consequence this model is not equivalent to the phase portrait of the ΛCDM model.

Structural stability

The idea of structural stability, emerged in the 1930's with the writings of Andronov, Leontovich and Pontryagin in Russia (they called it "roughly systems"). This idea is based on the observation that actual state of the system can never be specified exactly.

Among all dynamicists there is a shared prejudice that

1. there is a class of phase portraits that are far simpler than arbitrary ones which can explain why a considerable portion of the mathematical physics has been dominated by the search for the generic properties. The exceptional cases should not arise very often in application;
2. the physically realistic models of the world should posses some kind of structural stability because to have many dramatically different models all agreeing with observations would be fatal for the empirical methods of science.

The problem is how to define

1. a space of states and their equivalence,
2. a perturbation of the system.

The dynamical system is called structurally stable if all its δ-perturbations (sufficiently small) have an epsilon equivalent phase portrait. Therefore for the conception of structural stability we consider a δ perturbation of the vector field determined by right hand sides of the system which is small (measured by delta). We also need a conception of epsilon equivalence. This has the form of topological equivalence – a homeomorphism of the state space preserving the arrow of time on each trajectory.

Generic and non-generic phase portrait in ensemble

There is a simple way to introduce the metric in the space of all dynamical systems on the compactified plane. If $f \in C^1(\mathcal{M})$ where $\mathcal{M}$ is an open subset of $\mathbb{R}^n$, then the C^1 norm of f can be introduced in a standard way

$$||f||_1 = \sup_{x \in E} |f(x)| + \sup_{x \in E} \|Df(x)\|, \tag{97}$$

where $|\ldots|$ and $\|\ldots\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and the usual norm of the Jacobi matrix $Df(x)$, respectively. It is well known that the set of vectors field bounded in the C^1 norm forms a Banach space.

It is natural to use the defined norm to measure the distance between any two dynamical systems of the multiverse. If we consider some compact subset $\mathcal{K}$ of $\mathcal{M}$ then the C^1 norm of vector field f on $\mathcal{K}$ can be defined as

$$\|f\|_1 = \max_{x\in\mathcal{K}} |f(x)| + \max_{x\in\mathcal{K}} \|Df(x)\| < \infty. \tag{98}$$

Let $E = R^n$ then the ϵ-perturbation of f is the function $g \in C^1(\mathcal{M})$ form which $\|f - g\| < \epsilon$. The introduced language is suitable to reformulate the idea of structural stability given by Andronov and Pontryagin. The intuition is that f should be structurally stable vector field if for any vector field g near f, the vector fields f and g are topologically equivalent. A vector field $f \in C^1(\mathcal{M})$ is said to be structurally stable if there is an $\epsilon > 0$ such that for all $g \in C^1(\mathcal{M})$ with $\|f - g\|_1 < \epsilon$, f and g are topologically equivalent on open subsets of $\mathbb{R}^n$ called $\mathcal{M}$. To show that system is not structurally stable on $\mathbb{R}^n$ it is sufficient to show that f is not structurally stable on some compact $\mathcal{K}$ with nonempty interior.

The Sobolev metric introduced in the multiverse of dark energy models can be used to measure how far different cosmological model with dark energy are to the canonical ΛCDM model. For this aim let us consider a different dark energy models with dust matter and dark energy. We also for simplicity of presentation assume for all models have the same value of $\Omega_{\mathrm{m},0}$ parameters which can be obtained, for example, from independent extragalactic measurements. Then, the distance between any two cosmological model, say model '1' and model '2' is

$$d(1,\ 2) = \max_{x \in C}[|V_{1x} - V_{2x}|,\ |V_{1xx} - V_{2xx}|] \tag{99}$$

where we assumed the same value of the parameter H_0 measured at the present epoch for all cosmological models which we compare, V_1 and V_2 and their derivatives are only the parts of the potentials without the matter term.

Structural stability of planar dynamical systems

For planar dynamical systems (as is the case for the models under consideration) we have Peixoto's theorem.

Theorem

Structurally stable dynamical systems form open and dense subsets in the space of all dynamical systems defined on the compact manifold.

This theorem is a basic characterization of the structurally stable dynamical systems on the plane which offers the possibility of an exact definition of generic (typical) and non-generic (exceptional) cases (properties) employing the notion of structural stability. Unfortunately, there are no counterparts of this theorem in more dimensional cases when structurally unstable systems can also form open and dense subsets. For our aims it is important that Peixoto's theorem can characterize generic cosmological models in terms of the potential function.

The analysis of full dynamical behaviour of trajectories requires the study of the behaviour of trajectories at infinity, e.g. by means of the Poincaré sphere construction. We project the trajectories from centre of the unit sphere $S^2 = \{(X, Y, Z) \in \mathbf{R}^3 \colon X^2 + Y^2 + Z^2 = 1\}$ onto the (x, y) plane tangent to S^2 at either the north or south pole. Due to this central projection the critical points at infinity are spread out along the equator. Therefore if we project the upper hemisphere S^2 onto the (x, y) plane of dynamical system of the Newtonian type, then

$$x = \frac{X}{Z}, \quad y = \frac{Y}{Z}, \tag{100}$$

or

$$X = \frac{x}{\sqrt{1 + x^2 + \left(\frac{\partial V}{\partial x}\right)^2}}, \quad Y = \frac{y}{\sqrt{1 + x^2 + \left(\frac{\partial V}{\partial x}\right)^2}},$$
$$Z = \frac{1}{\sqrt{1 + x^2 + \left(\frac{\partial V}{\partial x}\right)^2}}. \tag{101}$$

While there is no counterpart of Peixoto's theorem in higher dimensions, it is easy to test whether a planar polynomial system has a structurally stable global phase portrait. In particular, a vector field on the Poincaré sphere will be structurally unstable if there is a non-hyperbolic critical point at infinity or if there is a trajectory connecting saddles on the equator of the Poincaré sphere S^2. In opposite case if additionally the number of critical points and limit cycles is finite, $\boldsymbol{f}$ is structurally stable on S^2. Following Peixoto's theorem the structural stability is a generic property of the C^1 vector fields on a compact two-dimensional differentiable manifold $\mathcal{M}$.

Let us introduce the following definition

Definition

If the set of all vector fields $\boldsymbol{f} \in C^r(\mathcal{M})$ $(r \geq 1)$ having a certain property contains an open dense subset of $C^r(\mathcal{M})$, then the property is called generic.

Generic and non-generic phase portrait on Poincaré sphere

If we consider some subclass of dark energy models described by the vector field $[y, -\partial V/\partial x]^T$ on the Poincaré sphere, then the right hand sides of the corresponding dynamical systems are of the polynomial form of degree m.

Theorem

Then f is structurally stable if

(i) *the number of critical points and limit cycles is finite and each critical point is hyperbolic . therefore a saddle point in finite domain,*

(ii) *there are no trajectories connecting saddle points.*

It is important that if the polynomial vector field f is structurally stable on the Poincaré sphere S^2 then the corresponding polynomial vector field $[y, -\partial V/\partial x]^T$ is structurally stable on $\mathbb{R}^2$.

Following Peixoto's theorem the structural stability is a generic property of C^1 vector fields on a compact two-dimensional differentiable manifold $\mathcal{M}$. If a vector field $f \in C^1(\mathcal{M})$ is not structurally stable it belongs to the bifurcation set $C^1(\mathcal{M})$. For such systems their global phase portrait changes as vector field passes through a point in the bifurcation set.

Therefore, in the class of dynamical systems on the compact manifold, the structurally stable systems are typical (generic) whereas structurally unstable are rather exceptional. In science modeling, both types of systems are used. While the structurally stable models describe "stable configuration" structurally unstable model can describe a fragile physical situation which requires fine tuning.

Structural stability of FRW cosmological models

From the physical point of view it is interesting to known whether a certain subset V of $C^r(\mathcal{M})$ (representing the class of cosmological accelerating models in our case) contains a dense subset because it means that this property (acceleration) is typical in V.
It is not difficult to establish some simple relation between the geometry of the potential function and localization of the critical points and its character for the case of dynamical systems of the Newtonian type:

1. The critical points of the systems under consideration $\dot{x} = y$, $y = -\frac{\partial V}{\partial x}$ lie always on the x axis, i.e. they represent static universes $y_0 = 0$, $x = x_0$;
2. The point $(x_0, 0)$ is a critical point of the Newtonian system iff it is a turning point of the potential function $V(x)$, i.e. $V(x) = E$ (E is the total energy of the system $E = y^2/2 + V(x)$; $E = 0$ for the case of flat models and $E = -k/2$ in general);

3. If $(x_0, 0)$ is a strict local maximum of $V(x)$, it is a saddle type critical point;
4. If $(x_0, 0)$ is a strict local minimum of the analytic function $V(x)$, it is a centre;
5. If $(x_0, 0)$ is a horizontal inflection point of the $V(x)$, it is a cusp;
6. The phase portraits of the Newtonian type systems have reflectional symmetry with respect to the y axis, i.e. $x \to x$, $y \to -y$.

All these properties are simple consequences of the Hartman-Grobman theorem

Theorem

Near the non-degenerate (hyperbolic) critical points the original dynamical system is equivalent to its linear part.

The character of a critical point is determined by the eigenvalues of the linearization matrix,

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{\partial^2 V}{\partial x^2} & 0 \end{bmatrix}_{(x_0,0)}$$

from the characteristic equation $\lambda^2 + \det A = 0$.
For a maximum of potential function we obtain a saddle with real λ_1, λ_2 of opposite signs, and for a minimum at the critical point we have a centre with $\lambda_{1,2}$ purely imaginary of mutually conjugate. Only if the potential function admits a local maximum at the critical point we have a structurally stable global phase portrait. Because $V \leq 0$ and $\partial V/\partial a = \frac{1}{6}(\rho(a) + 3p(a))a$ the Universe is decelerating if the strong energy condition is satisfied and accelerating if the strong energy condition is violated. Hence, among all simple scenarios, the one in which deceleration is followed by acceleration is the only structurally stable one.

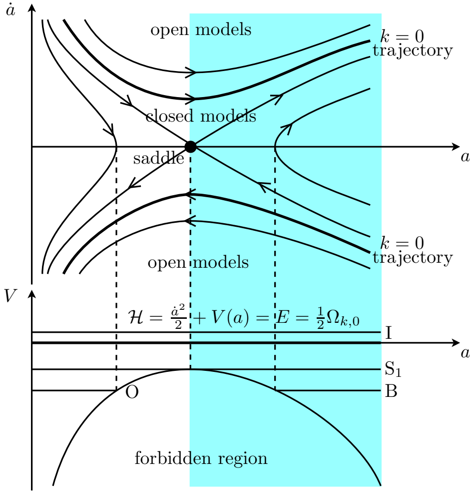

Figure: The model of an accelerating universe given in terms of the potential function and its phase portrait. The domain of acceleration is represented by shaded area. It is equivalent to the ΛCDM model scenario.

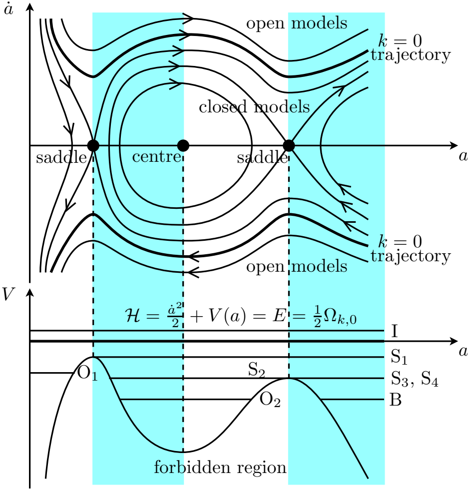

Figure: The model of an accelerating universe given in terms of the potential function and its phase portrait. The existence of two maxima induces structural instability of the system. It is the non-generic phase portrait for the universe accelerating in two domains.

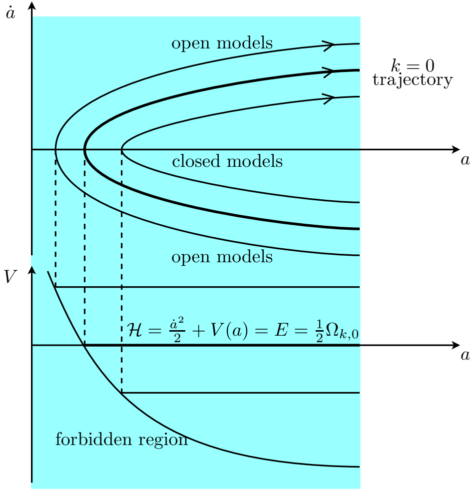

Figure: The model of an accelerating universe given in terms of the potential function and its phase portrait. The universe is accelerating for all trajectories. There is no critical point in the finite domain. While this system is structurally stable there is no matter dominating phase.

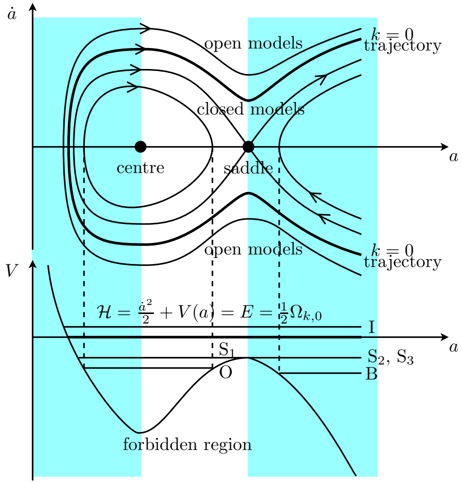

Figure: Bouncing models with the cosmological constant. There are three characteristic types of evolution: I – inflectional, O – oscillating, B – bouncing. The model is structurally unstable because of the presence of a non-hyperbolic critical point (center).

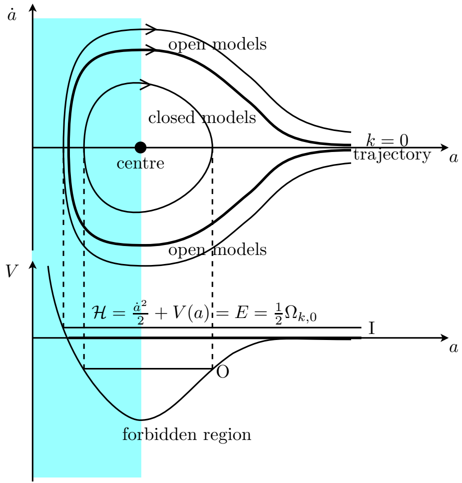

Figure: Bouncing models $H^2/H_0^2 = \Omega_{\mathrm{m},0}x^{-m} - \Omega_{n,0}x^{-n}$, $n > m$, $m, n = \text{const}$. They are closed for oscillating (O) models appearing for positive curvature, or open, representing bouncing flat and open cosmologies with a single bounce phase. It is structurally unstable.

We can see that there are two types of scenarios of cosmological models with matter dominated and dark energy dominated phases.

1. The ΛCDM scenario, where the early stage of evolution is dominated by both baryonic and dark matter, and late stages are described by the cosmological constant effects.
2. The bounce instead initial singularity squeezed into a cosmological scenario; one can distinguish cosmological models early bouncing phase of evolution (caused by the quantum bounce) from the classical bouncing models at which the expansion phase follows the contraction phase. In this paper by bouncing models we understand the models in the former sense (the modern one).

One can imagine different evolutional scenarios in terms of the potential function. Because of the existence of a bouncing phase which always gives rise to the presence of a non-hyperbolic critical point on the phase portrait one can conclude

- the bounce is not a generic property of the evolutional scenario,
- structural stability prefers the simplest evolutional scenario in which the deceleration epoch is followed by the acceleration phase.

The dynamical systems with the property of such switching rate of expansion, following the single-well potential are generic in the class of all dynamical systems on the plane.