Hawking radiation by Schwarzschild-de Sitter black holes: fermionic fields

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Outline

- This presentation is mainly based on our published paper:
  C.A. Sporea, A. Borowiec, *Int. J. Mod. Phys. D* **25** (2016) 1650043
- Introduction: Dirac eq. in curved spacetimes, Cartesian gauge
- Schwarzschild-de Sitter black holes
- Solutions to Dirac eq. in SdS geometry
- Analytical low energy SdS greybody factors
- Hawking radiation. Energy emission rate
- Conclusions

- The Dirac equation

$$\boxed{i\gamma^a D_a \psi - m\psi = 0}$$

it results from the gauge invariant action

$$\boxed{\mathcal{S} = \int \mathrm{d}^4x\,\sqrt{-g}\left\{\frac{i}{2}\overline{\psi}\gamma^a D_a\psi - \frac{i}{2}(\overline{D_a\psi})\gamma^a\psi - m\bar{\psi}\psi\right\}}$$

- The correct covariant derivative

$$\boxed{D_a = \partial_a + \frac{i}{2}S^{\,b}_{\ c}\,\omega^{\ c}_{ab}}$$

where $\partial_a = e^\mu_a \partial_\mu$, with e^μ_a the tetrad fields and $S^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$

- The spin-connection

$$\boxed{\omega^{\ c}_{ab} = e^\mu_a e^\nu_b\left(\hat{e}^c_\lambda \Gamma^\lambda_{\mu\nu} - \hat{e}^c_{\nu,\mu}\right)}$$

- The explicit form

$$\boxed{(i\gamma^a e_a^\mu \partial_\mu - m)\,\psi + \frac{i}{2}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,e_a^\mu)\gamma^a\psi - \frac{1}{4}\{\gamma^a, S^b{}_c\}\omega^{\ c}_{ab}\psi = 0}$$

- The tetrad fields $\hat{e}^a(x) = \hat{e}^a_\mu dx^\mu$ (i.e. the 1-forms) defining the Cartesian gauge are

$$\begin{aligned}
\hat{e}^0 &= h(r)dt \\
\hat{e}^1 &= \frac{1}{h(r)}\sin\theta\cos\phi\,dr + r\,\cos\theta\cos\phi\,d\theta - r\,\sin\theta\sin\phi\,d\phi \\
\hat{e}^2 &= \frac{1}{h(r)}\sin\theta\sin\phi\,dr + r\,\cos\theta\sin\phi\,d\theta + r\,\sin\theta\cos\phi\,d\phi \\
\hat{e}^3 &= \frac{1}{h(r)}\cos\theta\,dr - r\,\sin\theta\,d\theta
\end{aligned}$$

- Particle-like energy eigenspinors of positive frequency and energy E (I. I. Cotaescu, Mod. Phys. Lett. A 22, 2493, 2007)

$$\boxed{\begin{aligned}\psi(x) &= \psi_{E,j,m,\kappa}(t,r,\theta,\phi)\\ &= \frac{e^{-iEt}}{r\,h(r)^{1/2}}\left[F^{+}_{E,\kappa}(r)\Phi^{+}_{m_j,\kappa}(\theta,\phi)+F^{-}_{E,\kappa}(r)\Phi^{-}_{m_j,\kappa}(\theta,\phi)\right]\end{aligned}}$$

$F^{\pm}_{E,\kappa}(r)$ - radial wave functions.
$\Phi^{\pm}_{m_j,\kappa}(\theta,\phi)$ - usual four-component angular spinors.

- The antiparticle-like energy eigenspinors can be obtained directly using the charge conjugation as in the flat case:

$$\boxed{V_{E,j,m,\kappa} = (\psi_{E,j,m,\kappa})^{c} \equiv C(\bar{\psi}_{E,j,m,\kappa})^{T}, \quad C = i\gamma^{2}\gamma^{0}}$$

- The Schwarzschild-de Sitter line element

$$\boxed{ds^2 = h(r)\,dt^2 - \frac{dr^2}{h(r)} - r^2(d\theta^2 + \sin^2\theta\ d\phi^2)}$$

$$\boxed{h(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2}$$

- The radial Dirac equation for the upper component $F^+(r)$ reads:

$$\boxed{\frac{d^2F^+}{dx^2} + \left[\epsilon^2\left(\frac{1-\lambda\sqrt{h}}{1+\lambda\sqrt{h}}\right) + \frac{d}{dx}\left(\frac{k\sqrt{h}}{(1+\lambda\sqrt{h})r}\right) - \frac{k^2h}{(1+\lambda\sqrt{h})^2\,r^2}\right]F^+ = 0}$$

where: $\boxed{\dfrac{dr}{dx} = \dfrac{h}{1+\lambda\sqrt{h}}, \qquad \lambda = m/\epsilon}$

- For $r \to r_b$ and $r \to r_c$ the function $h(r) \to 0$ and we obtain a more simple equation

$$\boxed{\frac{d^2 F^+}{dx^2} + \epsilon^2 F^+ = 0}$$

having general solutions of the form

$$\boxed{F^+(x) = A\, e^{-i\epsilon x} + B\, e^{i\epsilon x}}$$

- Near the two horizons the new variable x behaves as:

$$\boxed{x \approx \begin{cases} \left(\frac{2M}{r_b^2} - \frac{2\Lambda}{3} r_b\right)^{-1} \ln h \equiv p \ln h, & \text{if } r \to r_b \\ \left(\frac{2M}{r_c^2} - \frac{2\Lambda}{3} r_c\right)^{-1} \ln h \equiv q \ln h, & \text{if } r \to r_c \end{cases}}$$

- After imposing the ingoing boundary condition at the black hole horizon the solution in this (transition) region becomes:

$$\boxed{F_b^+ = A^{tr} e^{-i\epsilon x} \approx A^{tr} e^{-i\epsilon p \ln h}}$$

- At the cosmological horizon we have no restrictions, thus the solution will be a combination of ingoing and outgoing modes:

$$\boxed{\begin{aligned} F_c^+ &= A^{in} e^{-i\epsilon x} + A^{out} e^{i\epsilon x} \\ &\approx A^{in} e^{-i\epsilon q \ln h} + A^{out} e^{i\epsilon q \ln h} \end{aligned}}$$

- In this intermediate region $r_b < r < r_c$ the radial equation reduces to:

$$\frac{d^2F^+}{dx^2} + \left[\frac{d}{dx}\left(\frac{k\sqrt{h}}{(1+\lambda\sqrt{h})r}\right) - \frac{k^2 h}{(1+\lambda\sqrt{h})^2\, r^2}\right] F^+ = 0$$

for which (after some calculations) we find the following solution:

$$F_I^+ = (A_2 + B_2\, C(r))\, F_{hom}^+$$

where

$$F_{hom}^+ = H_0^{-1}(1 - \frac{\Lambda}{3} I)$$

$$H_0 = C\left(\frac{1-\sqrt{h_0}}{1+\sqrt{h_0}}\right)^{-k}, \quad h_0 = 1 - \frac{2M}{r}$$

SdS black holes

and also

$$C(r)=\int\left(\frac{1-\sqrt{h_0}}{1+\sqrt{h_0}}\right)^{-2k}\left(\frac{1+\frac{\Lambda}{3}I}{1-\frac{\Lambda}{3}I}\right)\left(\frac{1}{h}+\frac{\lambda}{\sqrt{h}}\right)\frac{r^2\,dh}{2M-2\Lambda/3\,r^3}$$

$$I=\frac{1}{2}\int\frac{kr}{(\sqrt{h_0})^3}\,dr=\frac{1}{2}\frac{k}{r-2M}\Big[r\sqrt{h_0}(r^2+5M\,r-30M^2)$$
$$+15M^2(r-2M)\ln(2r\sqrt{h_0}+2r-2M)\Big]$$

- Near the black hole

$$\boxed{F_b^+ = A^{tr} e^{-i\epsilon x} \approx A^{tr} e^{-i\epsilon p \ln h}}$$

- in the intermediate region

$$\boxed{F_I^+ = (A_2 + B_2\, C(r))\, F_{hom}^+}$$

- near the cosmological horizon

$$\boxed{\begin{aligned} F_c^+ &= A^{in} e^{-i\epsilon x} + A^{out} e^{i\epsilon x} \\ &\approx A^{in} e^{-i\epsilon q \ln h} + A^{out} e^{i\epsilon q \ln h} \end{aligned}}$$

- At low energies our solutions behave as

$$\boxed{F_b^+ \approx A^{tr}(1 - i\epsilon p \ln h + ...)}$$

$$\boxed{F_c^+ \approx A^{in}(1 - i\epsilon q \ln h + ...) + A^{out}(1 + i\epsilon q \ln h + ...)}$$

$$\boxed{\lim_{r \to r_{b,c}} F_I^+ = \alpha_{b,c}\,(A_2 + \beta_{b,c}\, B_2 \ln h)}$$

$$\boxed{\begin{aligned} \alpha_{b,c} &= H_0^{-1}\left(1 - \frac{\Lambda}{3} I\right)\Bigg|_{r=r_{b,c}} \\ \beta_{b,c} &= \left(\frac{1-\sqrt{h_0}}{1+\sqrt{h_0}}\right)^{-2k}\left(\frac{1+\frac{\Lambda}{3}I}{1-\frac{\Lambda}{3}I}\right)\frac{r^2}{2M - 2\Lambda/3\, r^3}\Bigg|_{r=r_{b,c}} \end{aligned}}$$

- Matching of the solutions

$$\boxed{F_b^+ \approx A^{tr}(1 - i\epsilon p \ln h + \dots)}$$

$$\boxed{F_I^+ \approx \alpha_b \left(A_2 + \beta_b B_2 \ln h\right)}$$

- The result (1)

$$\boxed{A_2 = \frac{1}{\alpha_b} A^{tr} \qquad B_2 = -\frac{i\epsilon p}{\alpha_b \beta_b} A^{tr}}$$

- Matching of the solutions

$$F_I^+ \approx \alpha_b \left(A_2 + \beta_b B_2 \ln h\right)$$

$$F_c^+ \approx A^{in}(1 - i\epsilon q \ln h + \ldots) + A^{out}(1 + i\epsilon q \ln h + \ldots)$$

- The result (2)

$$A^{in} = \frac{\alpha_c}{2}\left(A_2 - \frac{\beta_c}{i\epsilon q} B_2\right) \qquad A^{out} = \frac{\alpha_c}{2}\left(A_2 + \frac{\beta_c}{i\epsilon q} B_2\right)$$

- The final result for the greybody factors (C.A. Sporea, A. Borowiec, IJMPD **25** (2016) 1650043)

$$\boxed{\Gamma_j(\epsilon) \equiv 1 - \left|\frac{A^{out}}{A^{in}}\right|^2 = 1 - \left(\frac{p\,\beta_c - q\,\beta_b}{p\,\beta_c + q\,\beta_b}\right)^2}$$

- Comparing numerically the greybody factors (or equivalently the absorption cross section) in the massless limit for the lowest angular quantum numbers ($j = 1/2$ for fermions, respectively $s = 0$ for scalars) we obtain that their ratio is approximatively.

$$\boxed{\frac{\Gamma_{j=\frac{1}{2}}}{\Gamma_{s=0}} \propto \frac{\sigma^{abs}_{j=\frac{1}{2}}}{\sigma^{abs}_{s=0}} \approx \frac{1}{12}}$$

- In the case of a Schwarzschild black hole the same ratio is equal with $1/8$

Λr_b^2	0.001	0.0025	0.005	0.0075	0.01
$\Gamma_{j=\frac{1}{2}}$	1.14	2.95	6.11	9.42	12.85
$\Gamma_{j=\frac{3}{2}}$	$1.38 \cdot 10^{-5}$	$9.07 \cdot 10^{-5}$	$3.84 \cdot 10^{-4}$	$9.03 \cdot 10^{-4}$	$1.67 \cdot 10^{-3}$
$\Gamma_{j=\frac{5}{2}}$	$1.2 \cdot 10^{-10}$	$1.94 \cdot 10^{-9}$	$1.56 \cdot 10^{-8}$	$5.9 \cdot 10^{-8}$	$1.2 \cdot 10^{-7}$

Table: The greybody factors for the first three modes (all the numerical values of $\Gamma_j(\epsilon)$ have been multiplied by a factor of 10^4).

- For each mode the value of Γ_j becomes higher as we increase the value of the cosmological constant Λ.
- The contribution of the lowest mode $j = 1/2$ to the emission spectra is the dominant one.
- These results are consistent with numerical calculations performed by S. F. Wu et al., Phys. Rev. D 78 (1998) 084010.

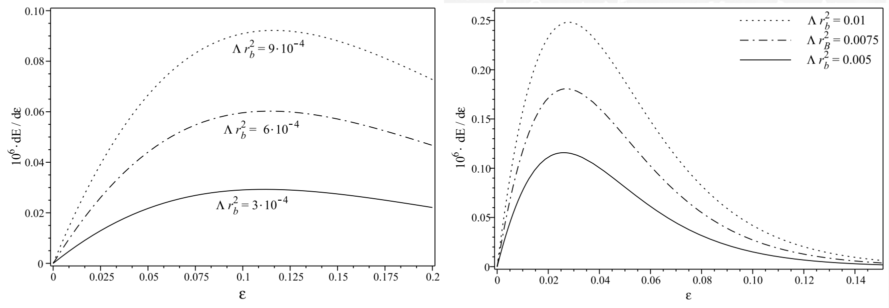

Figure: The fermion differential energy emission rate for different values of Λr_b^2. For the left panel we have set $r_b = 1$, respectively $r_b = 5$ for the right panel. *Obs: This spectra should be trusted for quantitative results only in the low energy regime.*

- The spectrum is enhanced with the increasing value of the cosmological constant;
- The energy emission rate for fermions vanishes in the limit $Energy \to 0$ (as in the case of asymptotically flat BHs);

Conclusions

- Deriving for the first time an analytical formula for low energy greybody factors for fermions emitted by a Schwarzschild-de Sitter black hole;
- For fermions the SdS greybody factors are constant for each mode at very low energies;
- However, for fermions Γ_j have a much more complicated dependence on r_b and r_c compared to the scalar case;
- The contribution of the lowest mode $j = 1/2$ to the emission spectra is the dominant one.
- The ratio fermions to scalars emitted by a SdS black hole is approx. $\frac{1}{12}$ (compared to $\frac{1}{8}$ for a Shw. BH).

The end

Thank you for your attention!

to infinity and beyond..

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