Non-Unitary Classical r-matrices and Gaudin Model with Boundary

Nenad Manojlović

Departamento de Matemática
Faculdade de Ciências e Tecnologia
Universidade do Algarve

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Outline

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Outline

Introduction

- In 1976 Michel Gaudin proposed the Hamiltonian describing "long-range" interacting spins in a chain. In 1982 he also introduce the so-called Gaudin algebras.
- In 1987 Sklyanin studied the rational model in the framework of the quantum inverse scattering method using the $s\ell(2)$ invariant classical r-matrix.
- Since then these models have attracted attention. Many interesting relations to different fields were established...
- Our interest is to study the non-periodic boundary conditions...

Outline

1. Introduction

2. Gaudin Model
   - Sklyanin's derivation in the periodic case
   - $s\ell(2)$ Gaudin model with boundary
   - Generalized Gaudin algebra
   - Algebraic Bethe Ansatz

Yang R-matrix

The XXX Heisenberg spin chain is related to the Yangian $\mathcal{Y}(s\ell(2))$ and the $SL(2)$-invariant Yang R-matrix

$$R(\lambda) = \lambda \mathbb{1} + \eta \mathcal{P} = \begin{pmatrix} \lambda+\eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix},$$

where λ is a spectral parameter, η is a quasi-classical parameter. We use $\mathbb{1}$ for the identity matrix and $\mathcal{P}$ for the permutation in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Periodic inhomogeneous XXX spin chain

We study an inhomogeneous XXX spin chain with N sites, with the local space $V_j = \mathbb{C}^{2s+1}$ and inhomogeneous parameter α_j. For simplicity, we start by considering periodic boundary conditions. The Hilbert space of the system is

$$\mathcal{H} = \bigotimes_{j=1}^{N} V_j = (\mathbb{C}^{2s+1})^{\otimes N}.$$

Lax operator

Following Sklyanin '87 we introduce the Lax operator

$$\mathbb{L}_{0m}(\lambda) = \mathbb{1} + \frac{\eta}{\lambda}\left(\vec{\sigma}_0 \cdot \vec{S}_m\right) = \frac{1}{\lambda}\begin{pmatrix} \lambda + \eta S^3_m & \eta S^-_m \\ \eta S^+_m & \lambda - \eta S^3_m \end{pmatrix},$$

here

$$S^\alpha_m = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1},$$

with $m = 1, 2, \ldots, N$ and S^α, $\alpha = 1, 2, 3$ are the spin operators

$$S^\alpha = \frac{1}{2}\sigma^\alpha = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & \delta_{\alpha 1} - \imath\delta_{\alpha 2} \\ \delta_{\alpha 1} + \imath\delta_{\alpha 2} & -\delta_{\alpha 3} \end{pmatrix}.$$

acting in $\mathbb{C}^2$. Notice that $\mathbb{L}(\lambda)$ is a two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$.

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acting in $\mathbb{C}^2$. Notice that $\mathbb{L}(\lambda)$ is a two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$.

Periodic inhomogeneous XXX spin chain

The so-called monodromy matrix

$$T(\lambda) = \mathbb{L}_{0N}(\lambda - \alpha_N) \cdots \mathbb{L}_{01}(\lambda - \alpha_1)$$

is used to describe the system. Notice that $T(\lambda)$ is a two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$.

RTT-relations

Due to the Yang-Baxter equation, it is straightforward to check that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda-\mu)T_0(\lambda)T_{0'}(\mu)=T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda-\mu).$$

The above equation is written in the tensor product of the auxiliary space $V_0\otimes V_{0'}=\mathbb{C}^2\otimes\mathbb{C}^2$.

Periodic transfer matrix

The trace of the monodromy matrix $T(\lambda)$ is the transfer matrix

$$t(\lambda) = \mathrm{tr}_0 \, T(\lambda).$$

The periodic boundary conditions and the RTT-relations imply that the transfer matrix at different values of the spectral parameter commute,

$$[t(\lambda), t(\mu)] = 0.$$

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Quantum determinant

The RTT-relations admit a central element

$$\Delta\left[T(\lambda)\right] = \mathrm{tr}_{00'} P^{-}_{00'} T_0\left(\lambda - \eta/2\right) T_{0'}\left(\lambda + \eta/2\right),$$

where

$$P^{-}_{00'} = \frac{\mathbb{1} - \mathcal{P}_{00'}}{2} = -\frac{R_{00'}\left(-\frac{\eta}{2}\right)}{2\eta}.$$

It is straightforward to show that

$$\left[\Delta\left[T(\mu)\right], T(\nu)\right] = 0.$$

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Expansion of the monodromy matrix

As the first step toward the study of the Gaudin model we consider the expansion of the monodromy matrix with respect to the quasi-classical parameter η

$$T(\lambda) = \mathbb{1} + \eta \sum_{m=1}^{N} \frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} + \frac{\eta^2}{2} \sum_{\substack{n,m=1 \\ n\neq m}}^{N} \frac{\mathbb{1}_0 \left(\vec{S}_m \cdot \vec{S}_n \right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)}$$

$$+ \frac{\eta^2}{2} \sum_{m=1}^{N} \left(\sum_{n>m}^{N} \frac{\imath \vec{\sigma}_0 \cdot \left(\vec{S}_n \times \vec{S}_m \right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \sum_{n<m}^{N} \frac{\imath \vec{\sigma}_0 \cdot \left(\vec{S}_m \times \vec{S}_n \right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} \right) + \mathcal{O}(\eta^3).$$

Gaudin Lax operator

If the Gaudin Lax matrix is defined by

$$L_0(\lambda) = \sum_{m=1}^{N} \frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m}$$

and the quasi-classical property of the Yang R-matrix

$$\frac{1}{\lambda} R(\lambda) = \mathbb{1} - \eta r(\lambda), \qquad \text{where} \qquad r(\lambda) = -\frac{\mathcal{P}}{\lambda}$$

is taken into account, then substitution of the expansion of the monodromy matrix into the RTT-relations yields the so-called Sklyanin linear bracket

$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)] .$$

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$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)] \, .$$

Expansion of the transfer matrix

Also, using the expansion of the monodromy matrix it is evident that

$$t(\lambda) = 2 + \eta^2 \sum_{m=1}^{N} \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \mathcal{O}(\eta^3).$$

Expansion of the transfer matrix monodromy matrix

Analogously, we obtain the expansion of the quantum determinant

$$\Delta\left[T(\lambda)\right] = \mathbb{1} + \eta^2 \left(\sum_{m=1}^{N} \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{(\lambda - \alpha_m)(\lambda - \alpha_n)} - \frac{1}{2} \mathrm{tr}\, L^2(\lambda) \right) + \mathcal{O}(\eta^3).$$

Generation function of the Gaudin Hamiltonians

To obtain the generation function of the Gaudin Hamiltonians notice that

$$t(\lambda) - \Delta\left[T(\lambda)\right] = \mathbb{1} + \frac{\eta^2}{2} \mathrm{tr}_0\, L^2(\lambda) + \mathcal{O}(\eta^3),$$

and by definition we have

$$\left[\left(t(\lambda) - \Delta\left[T(\lambda)\right]\right), \left(t(\mu) - \Delta\left[T(\mu)\right]\right)\right] = 0.$$

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Generation function of the Gaudin Hamiltonians

Therefore

$$\tau(\lambda) = \frac{1}{2} \mathrm{tr}_0 \, L^2(\lambda)$$

commute for different values of the spectral parameter,

$$[\tau(\lambda), \tau(\mu)] = 0.$$

Generation function of the Gaudin Hamiltonians

Moreover, the partial fraction decomposition of the generating function is

$$\tau(\lambda) = \sum_{m=1}^{N} \frac{2H_m}{\lambda - \alpha_m} + \sum_{m=1}^{N} \frac{\vec{S}_m \cdot \vec{S}_m}{(\lambda - \alpha_m)^2},$$

and the Gaudin Hamiltonians, in the periodic case, are

$$H_m = \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n}.$$

This shows that $\tau(\lambda)$ is the generating function of the Gaudin Hamiltonians in the periodic case (Sklyanin '87).

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K-matrix for the Gaudin model

For the study of the open Gaudin model it is necessary to impose

$$\left(\lim_{\eta\to 0} K^{+}(\lambda)\right) K^{-}(\lambda) = \left(\xi^2 - \lambda^2 (1 + \phi\psi)\right) \mathbb{1}.$$

In particular, this implies that the parameters of the reflection matrices on the left and on the right end of the chain are the same. In general this not the case in the study of the open spin chain. However, this condition is essential for the Gaudin model.

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Sklyanin Monodromy matrix $\mathcal{T}(\lambda)$

We use the Sklyanin approach to integrable spin chains with non-periodic boundary conditions. The Sklyanin monodromy matrix $\mathcal{T}(\lambda)$ is

$$\mathcal{T}_0(\lambda) = T_0(\lambda) K_0^-(\lambda) \widetilde{T}_0(\lambda),$$

it consists of the matrix $T(\lambda)$, a reflection matrix $K^-(\lambda)$ and the matrix

$$\widetilde{T}_0(\lambda) = \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix} = \mathbb{L}_{01}(\lambda + \alpha_1 + \eta) \cdots \mathbb{L}_{0N}(\lambda + \alpha_N + \eta).$$

Sklyanin Monodromy matrix $\mathcal{T}(\lambda)$

By construction, the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$ are

$$R_{00'}(\lambda-\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda+\mu)\mathcal{T}_{0'}(\mu) = \mathcal{T}_{0'}(\mu)R_{00'}(\lambda+\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda-\mu).$$

Monodromy matrix $\mathcal{T}(\lambda)$

From the equation above we can read off the commutation relations of the entries of the monodromy matrix

$$\mathcal{T}(\lambda) = \left(\begin{array}{cc} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{array} \right).$$

Sklyanin determinant

These exchange relations admit a central element, the so-called Sklyanin determinant,

$$\Delta\left[\mathcal{T}(\lambda)\right] = \mathrm{tr}_{00'} P^{-}_{00'} \mathcal{T}_0(\lambda - \eta/2) R_{00'}(2\lambda) \mathcal{T}_{0'}(\lambda + \eta/2).$$

Generating function of the Gaudin Hamiltonians with boundary terms

In order to simplify some formulae we introduce the notation

$$K^{-}(\lambda) = K(\lambda)$$
$$\mathcal{L}(\lambda) = L_0(\lambda) - K_0(\lambda) L_0(-\lambda) K_0^{-1}(\lambda).$$

Finally, the expansion reads

$$2\lambda t(\lambda) - \Delta\left[\mathcal{T}(\lambda)\right] = 2\lambda\left(\xi^2 - \lambda^2(1+\phi\psi)\right) + \eta\left(\xi^2 - 3\lambda^2(1+\phi\psi)\right) + \eta^2\lambda\left(\xi^2 - \lambda^2(1+\phi\psi)\right)\operatorname{tr}_0 \mathcal{L}_0^2(\lambda) - \frac{\eta^2\lambda}{2}(1+\phi\psi) + \mathcal{O}(\eta^3).$$

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Generating function of the Gaudin Hamiltonians with boundary terms

... in the trigonometric case the expansion reads

$$t(\lambda) - \frac{\Delta\left[\mathcal{T}(\lambda)\right]}{\sinh(2\lambda)} = \det K_0(\lambda) + \eta \left(\mathrm{tr}_0 K_0'(-\lambda) K_0(\lambda) + \mathrm{tr}_{00'} P_{00'}^{-} K_0(\lambda) r_{00'}(2\lambda) K_{0'}(\lambda)\right)$$
$$+ \frac{\eta^2}{2} \det K_0(\lambda)\, \mathrm{tr}_0\, \mathcal{L}_0^2(\lambda) + \frac{\eta^2}{2} \left(\mathrm{tr}_0 K_0''(-\lambda) K_0(\lambda) - \frac{1}{4}\, \mathrm{tr}_0 K_0''(\lambda) K_0(-\lambda)\right.$$
$$\left. + \frac{1}{2}\, \mathrm{tr}_{00'} P_{00'}^{-} K_0'(\lambda) K_{0'}'(\lambda) - \frac{1}{\sinh(2\lambda)}\, \mathrm{tr}_{00'} P_{00'}^{-} K_0(\lambda) \partial_\eta^2 R_{00'}(2\lambda)\big|_{\eta=0} K_{0'}(\lambda)\right) + \mathcal{O}(\eta^3).$$

Generating function of the Gaudin Hamiltonians with boundary terms

This shows that

$$\tau(\lambda) = \mathrm{tr}_0\, \mathcal{L}_0^2(\lambda)$$

commute for different values of the spectral parameter,

$$[\tau(\lambda), \tau(\mu)] = 0.$$

and therefore can be considered to be the generating function of Gaudin Hamiltonians with boundary terms.

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Lax operator for the Gaudin model with boundary

With the aim of obtaining the Gaudin Hamiltonians with the boundary terms from the generating function, it is instructive to study the representation of $\mathcal{L}_0(\lambda)$ in terms of the local spin operators

$$\mathcal{L}_0(\lambda) = \sum_{m=1}^{N} \left(\frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} + \frac{\vec{\sigma}_0 \cdot \left(K_m^{-1}(\lambda) \vec{S}_m K_m(\lambda) \right)}{\lambda + \alpha_m} \right) .$$

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Generating function of the Gaudin Hamiltonians with boundary terms

It is straightforward to obtain the expression for the generating function in terms of the local operators

$$\tau(\lambda) = 2\sum_{m,n=1}^{N}\left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{\vec{S}_m\cdot\left(K_n^{-1}(\lambda)\vec{S}_nK_n(\lambda)\right)+\left(K_n^{-1}(\lambda)\vec{S}_nK_n(\lambda)\right)\cdot\vec{S}_m}{(\lambda-\alpha_m)(\lambda+\alpha_n)} + \frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right).$$

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Gaudin Hamiltonians with boundary terms

The Gaudin Hamiltonians with the boundary terms are obtained from the partial fraction decomposition of the generating function. They correspond, up to constant multiplicative factors and constant, scalar additive terms, to the residues of the of the generating function at the poles $\lambda = \alpha_m$

$$H_m = \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \frac{1}{2}\sum_{n=1}^{N} \frac{\left(K_m(\alpha_m)\vec{S}_m K_m^{-1}(\alpha_m)\right)\cdot \vec{S}_n + \vec{S}_n \cdot \left(K_m(\alpha_m)\vec{S}_m K_m^{-1}(\alpha_m)\right)}{\alpha_m + \alpha_n},$$

Gaudin Hamiltonians with boundary terms

and $\lambda = -\alpha_m$

$$\widetilde{H}_m = \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n}$$

$$+\frac{1}{2} \sum_{n=1}^{N} \frac{\left(K_m(-\alpha_m)\vec{S}_n K_m^{-1}(-\alpha_m)\right) \cdot \vec{S}_n + \vec{S}_n \cdot \left(K_m(-\alpha_m)\vec{S}_m K_m^{-1}(-\alpha_m)\right)}{\alpha_m + \alpha_n}.$$

Gaudin Hamiltonians with boundary terms

... in the trigonometric case we have

$$H_m = \sum_{n\neq m}^{N}\left(\coth(\alpha_m-\alpha_n)\,S_m^3S_n^3 + \frac{S_m^+S_n^- + S_m^-S_n^+}{2\sinh(\alpha_m-\alpha_n)}\right) + \sum_{n=1}^{N}\coth(\alpha_m+\alpha_n)\,\frac{S_m^3S_n^3+S_n^3S_m^3}{2}$$

$$+\frac{\psi}{\kappa}\,\frac{\sinh(2\alpha_m)}{\sinh(\xi+\alpha_m)}\sum_{n=1}^{N}\frac{S_m^3S_n^+ + S_n^+S_m^3}{2\sinh(\alpha_m+\alpha_n)} + \frac{\sinh(\xi-\alpha_m)}{2\sinh(\xi+\alpha_m)}\sum_{n=1}^{N}\frac{S_m^-S_n^+ + S_n^+S_m^-}{2\sinh(\alpha_m+\alpha_n)}$$

$$-\frac{\psi}{\kappa}\,\frac{\sinh(2\alpha_m)}{\sinh(\xi-\alpha_m)}\sum_{n=1}^{N}\coth(\alpha_m+\alpha_n)\,\frac{S_m^+S_n^3+S_n^3S_m^+}{2} + \frac{\sinh(\xi+\alpha_m)}{2\sinh(\xi-\alpha_m)}\sum_{n=1}^{N}\frac{S_m^+S_n^- + S_n^-S_m^+}{2\sinh(\alpha_m+\alpha_n)}$$

$$-\frac{\psi^2}{\kappa^2}\,\frac{\sinh^2(2\alpha_m)}{2\sinh(\xi-\alpha_m)\sinh(\xi+\alpha_m)}\sum_{n=1}^{N}\frac{S_m^+S_n^+ + S_n^+S_m^+}{2\sinh(\alpha_m+\alpha_n)}.$$

Outline

Reflection equation

In 1988 Sklyanin introduce non-periodic boundary conditions compatible with the integrability of the system.
Boundary conditions on the left and right sites of the chain are encoded in the left and right reflection matrices K^- and K^+.
The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation. It is written in the following form for the left reflection matrix $K^-(\lambda) \in \mathrm{End}(\mathbb{C}^2)$

$$R_{12}(\lambda-\mu)K_1^-(\lambda)R_{21}(\lambda+\mu)K_2^-(\mu) = K_2^-(\mu)R_{12}(\lambda+\mu)K_1^-(\lambda)R_{21}(\lambda-\mu).$$

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Quasi-classical limit of the RE

Our aim is to derive the commutation relations between the entries of the Lax operator as the quasi-classical limit of the exchange relations above. As the first step in this direction we observe

$$(\mathbb{1} - \eta r_{12}(\lambda - \mu))\, K_1(\lambda)\, (\mathbb{1} - \eta r_{21}(\lambda + \mu))\, K_2(\mu) =$$
$$= K_2(\mu)\, (\mathbb{1} - \eta r_{12}(\lambda + \mu))\, K_1(\lambda)\, (\mathbb{1} - \eta r_{21}(\lambda - \mu))$$

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$$= K_2(\mu) \left(\mathbb{1} - \eta r_{12}(\lambda + \mu)\right) K_1(\lambda) \left(\mathbb{1} - \eta r_{21}(\lambda - \mu)\right)$$

Classical RE

Since the condition obtained at the zero order in η of the above equation is identically satisfied for any $K(\lambda)$, dividing by η and taking the limit $\eta \to 0$ we obtain the classical reflection equation

$$r_{12}(\lambda-\mu)K_1(\lambda)K_2(\mu)+K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu)=$$
$$=K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda)+K_2(\mu)K_1(\lambda)r_{21}(\lambda-\mu).$$

Quadratic identity

The terms of the second order in η are

$$r_{12}(\lambda-\mu)K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu) = K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda)r_{21}(\lambda-\mu).$$

This equation is also satisfied by the K-matrix and the classical r-matrix.

Classical YBE

Let us recall that the classical r-matrix has the unitarity property

$$r_{21}(-\lambda) = -r_{12}(\lambda),$$

and satisfies the classical Yang-Baxter equation

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda - \mu), r_{13}(\lambda) + r_{23}(\mu)] = 0.$$

Quasi-classical limit of the ER

Now we can proceed to the derivation of the relevant linear bracket relations of the Lax operator. The desired relations can be obtained by writing the exchange relations above in the following form

$$
\begin{aligned}
&\left(\mathbb{1} - \eta r_{00'}(\lambda-\mu)\right) \mathcal{T}_0(\lambda) \left(\mathbb{1} - \eta r_{0'0}(\lambda+\mu)\right) \mathcal{T}_{0'}(\mu) = \\
&= \mathcal{T}_{0'}(\mu) \left(\mathbb{1} - \eta r_{00'}(\lambda+\mu)\right) \mathcal{T}_0(\lambda) \left(\mathbb{1} - \eta r_{0'0}(\lambda-\mu)\right)
\end{aligned}
$$

Quasi-classical limit of the ER

and substituting the expansion of $\mathcal{T}(\lambda)$ in powers of η

$$\mathcal{T}(\lambda) = K(\lambda) + \eta\,\mathcal{L}(\lambda)K(\lambda) + \frac{\eta^2}{2}\,\frac{d^2\mathcal{T}(\lambda)}{d\eta^2}|_{\eta=0} + \mathcal{O}(\eta^3).$$

The zero and first orders in η are identically satisfied for the matrix $K(\lambda)$. The relations we seek follow from the terms of the second order in η. When the terms containing the second order derivatives of $\mathcal{T}$ are eliminated and the quadratic identity above is used to eliminate the other two terms, there are ten terms remaining.

Quasi-classical limit of the ER

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Linear bracket

Then using twice the classical reflection equation and the unitarity property and multiplying both sides from the right by $K_0^{-1}(\lambda)K_{0'}^{-1}(\mu)$ one obtains

$$\left[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)\right] = \left[r_{00'}(\lambda-\mu) - K_{0'}(\mu) r_{00'}(\lambda+\mu) K_{0'}^{-1}(\mu), \mathcal{L}_0(\lambda)\right] - \left[r_{0'0}(\mu-\lambda) - K_0(\lambda) r_{0'0}(\mu+\lambda) K_0^{-1}(\lambda), \mathcal{L}_{0'}(\mu)\right].$$

Linear bracket

Defining

$$r^K_{00'}(\lambda,\mu) = r_{00'}(\lambda-\mu) - K_{0'}(\mu) r_{00'}(\lambda+\mu) K^{-1}_{0'}(\mu),$$

the commutation relations are given by

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = \left[r^K_{00'}(\lambda,\mu), \mathcal{L}_0(\lambda)\right] - \left[r^K_{0'0}(\mu,\lambda), \mathcal{L}_{0'}(\mu)\right].$$

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Generalized classical Yang-Baxter equation

The linear bracket above is obviously anti-symmetric. It obeys the Jacobi identity because the r-matrix r^K satisfies the so-calledgeneralized classical Yang-Baxter equation

$$[r^K_{32}(\nu,\mu), r^K_{13}(\lambda,\nu)] + [r^K_{12}(\lambda,\mu), r^K_{13}(\lambda,\nu) + r^K_{23}(\mu,\nu)] = 0.$$

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Local realization of the Lax operator

The local realization of the Lax operator of the non-periodic $s\ell(2)$ Gaudin model is given by

$$\mathcal{L}_0(\lambda) = \begin{pmatrix} H(\lambda) & F(\lambda) \\ E(\lambda) & -H(\lambda) \end{pmatrix}$$

$$= \sum_{m=1}^{N} \left(\frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} + \frac{K_0(\lambda)\vec{\sigma}_0 K_0^{-1}(\lambda) \cdot \vec{S}_m}{\lambda + \alpha_m} \right),$$

Local realization of the Lax operator

with the following local realization for its entries

$$E(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^+}{\lambda - \alpha_m} + \frac{(\xi + \lambda\nu) S_m^+}{(\xi - \lambda\nu)(\lambda + \alpha_m)} \right),$$

$$H(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^3}{\lambda - \alpha_m} + \frac{\lambda\psi\, S_m^+ + (\xi - \lambda\nu) S_m^3}{(\xi - \lambda\nu)(\lambda + \alpha_m)} \right),$$

$$F(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^-}{\lambda - \alpha_m} + \frac{(\xi - \lambda\nu)^2 S_m^- - \lambda^2\psi^2 S_m^+ - 2\lambda\psi(\xi - \lambda\nu) S_m^3}{(\xi + \lambda\nu)(\xi - \lambda\nu)(\lambda + \alpha_m)} \right).$$

Generalized $s\ell(2)$ Gaudin algebra

The linear bracket based on the r-matrix $r^K_{00'}(\lambda, \mu)$ defines the Lie algebra the so-called generalized $s\ell(2)$ Gaudin algebra.

It is instructive to introduce the generators $\widetilde{e}(\lambda)$, $\widetilde{h}(\lambda)$ and $\widetilde{f}(\lambda)$ as the following linear combinations of the original ones

$$\widetilde{e}(\lambda) = \frac{-\xi + \lambda\nu}{\lambda} E(\lambda),$$

$$\widetilde{h}(\lambda) = \frac{1}{\lambda}\left(H(\lambda) - \frac{\psi\lambda}{2\xi} E(\lambda)\right),$$

$$\widetilde{f}(\lambda) = \frac{1}{\lambda}\left((\xi + \lambda\nu)F(\lambda) + \psi\lambda H(\lambda)\right).$$

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Generalized $s\ell(2)$ Gaudin algebra

Notably in this basis the following three relations vanish

$$\left[\widetilde{e}(\lambda), \widetilde{e}(\mu)\right] = \left[\widetilde{h}(\lambda), \widetilde{h}(\mu)\right] = \left[\widetilde{f}(\lambda), \widetilde{f}(\mu)\right] = 0.$$

Therefore there are only three nontrivial relations

$$\left[\widetilde{h}(\lambda), \widetilde{e}(\mu)\right] = \frac{2}{\lambda^2 - \mu^2}\left(\widetilde{e}(\mu) - \widetilde{e}(\lambda)\right),$$

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Generalized $s\ell(2)$ Gaudin algebra

$$\left[\tilde{h}(\lambda), \tilde{f}(\mu)\right] = \frac{-2}{\lambda^2-\mu^2}\left(\tilde{f}(\mu)-\tilde{f}(\lambda)\right) - \frac{2\psi\nu}{(\lambda^2-\mu^2)\xi}\left(\mu^2\tilde{h}(\mu)-\lambda^2\tilde{h}(\lambda)\right) - \frac{\psi^2}{(\lambda^2-\mu^2)\xi^2}\left(\mu^2\tilde{e}(\mu)-\lambda^2\tilde{e}(\lambda)\right),$$

$$\left[\tilde{e}(\lambda), \tilde{f}(\mu)\right] = \frac{2\psi\nu}{(\lambda^2-\mu^2)\xi}\left(\mu^2\tilde{e}(\mu)-\lambda^2\tilde{e}(\lambda)\right) - \frac{4}{\lambda^2-\mu^2}\left((\xi^2-\mu^2\nu^2)\tilde{h}(\mu)-(\xi^2-\lambda^2\nu^2)\tilde{h}(\lambda)\right).$$

Generalized $s\ell(2)$ Gaudin algebra

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Generalized $s\ell(2)$ Gaudin algebra

With the aim of simplifying further the relations above we introduce the new generators $e(\lambda), h(\lambda)$ and $f(\lambda)$ as the following linear combinations of the previous ones

$$e(\lambda) = \widetilde{e}(\lambda),$$

$$h(\lambda) = \widetilde{h}(\lambda) + \frac{\psi}{2\xi\nu}\widetilde{e}(\lambda),$$

$$f(\lambda) = \widetilde{f}(\lambda) + \frac{\psi\xi}{\nu}\widetilde{h}(\lambda) + \frac{\psi^2}{4\nu^2}\widetilde{e}(\lambda).$$

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\end{aligned}
$$

Generalized $s\ell(2)$ Gaudin algebra

It is straightforward to check that in the new basis we continue to have

$$[e(\lambda), e(\mu)] = [h(\lambda), h(\mu)] = [f(\lambda), f(\mu)] = 0.$$

Generalized $s\ell(2)$ Gaudin algebra

But the key simplification occurs in the three nontrivial relations which are now given by

$$[h(\lambda), e(\mu)] = \frac{2}{\lambda^2 - \mu^2} \left(e(\mu) - e(\lambda) \right),$$

$$[h(\lambda), f(\mu)] = \frac{-2}{\lambda^2 - \mu^2} \left(f(\mu) - f(\lambda) \right),$$

$$[e(\lambda), f(\mu)] = \frac{-4}{\lambda^2 - \mu^2} \left((\xi^2 - \mu^2 \nu^2)\, h(\mu) - (\xi^2 - \lambda^2 \nu^2)\, h(\lambda) \right).$$

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$$[e(\lambda), f(\mu)] = \frac{-4}{\lambda^2 - \mu^2} \Big((\xi^2 - \mu^2\nu^2)\, h(\mu) - (\xi^2 - \lambda^2\nu^2)\, h(\lambda) \Big).$$

Generalized trigonometric $s\ell(2)$ Gaudin algebra

In some sense the situation is very similar to the rational case. The local local realization for the entries of the Lax matrix now reads

$$E(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^+}{\sinh(\lambda - \alpha_m)} + \frac{\sinh(\xi - \lambda)\, S_m^+}{\sinh(\xi + \lambda)\sinh(\lambda + \alpha_m)} \right),$$

$$H(\lambda) = \sum_{m=1}^{N} \left(\coth(\lambda - \alpha_m)\, S_m^3 + \coth(\lambda + \alpha_m)\, S_m^3 + \frac{\psi \sinh(2\lambda)\, S_m^+}{\kappa \sinh(\xi + \lambda)\sinh(\lambda + \alpha_m)} \right),$$

$$F(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^-}{\sinh(\lambda - \alpha_m)} + \frac{\sinh(\xi + \lambda)\, S_m^-}{\sinh(\xi - \lambda)\sinh(\lambda + \alpha_m)} - \frac{2\psi \sinh(2\lambda)}{\kappa \sinh(\xi - \lambda)} \coth(\lambda + \alpha_m)\, S_m^3 \right.$$
$$\left. - \frac{\psi^2 \sinh^2(2\lambda)\, S_m^+}{\kappa^2 \sinh(\xi - \lambda)\sinh(\xi + \lambda)\sinh(\lambda + \alpha_m)} \right).$$

Generalized $s\ell(2)$ Gaudin algebra

Unfortunately, the final key simplification does not exist in this case so the formulae of the commutators are complicated

$$[h(\lambda), e(\mu)] = \frac{1}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)} \left(e(\mu) - e(\lambda)\right),$$

$$[h(\lambda), f(\mu)] = \frac{-1}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)} \left(f(\mu) - f(\lambda)\right) + \frac{2\psi \coth(\xi)}{\kappa \sinh(\lambda-\mu)\sinh(\lambda+\mu)} \times$$
$$\times \left(\sinh^2(\mu)\, h(\mu) - \sinh^2(\lambda)\, h(\lambda)\right) + \frac{2\psi^2}{\kappa^2 \sinh(\lambda-\mu)\sinh(\lambda+\mu)\sinh^2(\xi)} \times$$
$$\times \left(\sinh^2(\mu)\, e(\mu) - \sinh^2(\lambda)\, e(\lambda)\right),$$

$$[e(\lambda), f(\mu)] = \frac{-2\psi \coth(\xi)}{\kappa \sinh(\lambda-\mu)\sinh(\lambda+\mu)} \left(\sinh^2(\mu)\, e(\mu) - \sinh^2(\lambda)\, e(\lambda)\right) + \frac{2}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)} \times$$
$$\times \left(\sinh(\xi-\mu)\sinh(\xi+\mu)\, h(\mu) - \sinh(\xi-\lambda)\sinh(\xi+\lambda)\, h(\lambda)\right).$$

Outline

Algebraic Bethe Ansatz – $s\ell(2)$ Gaudin with boundary

In every $V_m = \mathbb{C}^{2s+1}$ there exists a vector $\omega_m \in V_m$ such that

$$S_m^3 \, \omega_m = s_m \, \omega_m \quad \text{and} \quad S_m^+ \, \omega_m = 0.$$

Consequently we define a vector Ω_+ to be

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}.$$

It follows that

$$e(\lambda)\Omega_+ = 0 \quad \text{and} \quad h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+,$$

with

$$\rho(\lambda) = \frac{1}{\lambda} \sum_{m=1}^{N} \left(\frac{s_m}{\lambda - \alpha_m} + \frac{s_m}{\lambda + \alpha_m} \right) = \sum_{m=1}^{N} \frac{2 s_m}{\lambda^2 - \alpha_m^2}.$$

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Algebraic Bethe Ansatz – $s\ell(2)$ Gaudin with boundary

The vector Ω_+ is an eigenvector of the generating function $\tau(\lambda)$

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+ = 2\lambda^2 \left(\rho^2(\lambda) + \frac{2\nu^2\,\rho(\lambda)}{\xi^2 - \lambda^2\nu^2} - \frac{\rho'(\lambda)}{\lambda}\right)\Omega_+.$$

The Bethe vector $\varphi_1(\mu)$

The first Bethe vector is given by

$$\varphi_1(\mu) = f(\mu)\Omega_+.$$

The off-shell action of the generating function $\tau(\lambda)$ on $\varphi_1(\mu)$ is

$$\tau(\lambda)\varphi_1(\mu) = \chi_1(\lambda,\mu)\varphi_1(\mu) + \frac{8\lambda^2(\xi^2-\mu^2\nu^2)}{(\lambda^2-\mu^2)(\xi^2-\lambda^2\nu^2)}\left(\rho(\mu)+\frac{\nu^2}{\xi^2-\mu^2\nu^2}\right)\varphi_1(\lambda),$$

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The Bethe vector $\varphi_1(\mu)$

with

$$\chi_1(\lambda,\mu) = \chi_0(\lambda) - \frac{8\lambda^2}{\lambda^2-\mu^2}\left(\rho(\lambda) + \frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right).$$

The unwanted term vanishes when the following Bethe equation is imposed on the parameter μ,

$$\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2} = 0.$$

The Bethe vector $\varphi_2(\mu_1, \mu_2)$

The Bethe vector $\varphi_2(\mu_1, \mu_2)$ is the following symmetric function of its arguments

$$\varphi_2(\mu_1, \mu_2) = f(\mu_1) f(\mu_2) \Omega_+ .$$

The off-shell action of the generating function $\tau(\lambda)$ on $\varphi_2(\mu_1, \mu_2)$ reads

$$\begin{aligned}
\tau(\lambda)\varphi_2(\mu_1, \mu_2) &= \chi_2(\lambda, \mu_1, \mu_2)\varphi_2(\mu_1, \mu_2) \\
&+ \frac{8\lambda^2}{\lambda^2 - \mu_1^2} \frac{\xi^2 - \mu_1^2\nu^2}{\xi^2 - \lambda^2\nu^2} \left(\rho(\mu_1) + \frac{\nu^2}{\xi^2 - \mu_1^2\nu^2} - \frac{2}{\mu_1^2 - \mu_2^2} \right) \varphi_2(\lambda, \mu_2) \\
&+ \frac{8\lambda^2}{\lambda^2 - \mu_2^2} \frac{\xi^2 - \mu_2^2\nu^2}{\xi^2 - \lambda^2\nu^2} \left(\rho(\mu_2) + \frac{\nu^2}{\xi^2 - \mu_2^2\nu^2} - \frac{2}{\mu_2^2 - \mu_1^2} \right) \varphi_2(\mu_1, \lambda),
\end{aligned}$$

The Bethe vector $\varphi_2(\mu_1, \mu_2)$

with the eigenvalue

$$\chi_2(\lambda, \mu_1, \mu_2) = \chi_0(\lambda) - \sum_{i=1}^{2} \frac{8\lambda^2}{\lambda^2 - \mu_i^2} \left(\rho(\lambda) + \frac{\nu^2}{\xi^2 - \lambda^2\nu^2} - \frac{1}{\lambda^2 - \mu_{3-i}^2} \right).$$

The two unwanted terms in the action above are canceled when the following Bethe equations are imposed on the parameters μ_1 and μ_2,

$$\rho(\mu_1) + \frac{\nu^2}{\xi^2 - \mu_1^2\nu^2} - \frac{2}{\mu_1^2 - \mu_2^2} = 0,$$

$$\rho(\mu_2) + \frac{\nu^2}{\xi^2 - \mu_2^2\nu^2} - \frac{2}{\mu_2^2 - \mu_1^2} = 0.$$

ABA – trigonometric $s\ell(2)$ Gaudin with boundary

Here again we have that

$$e(\lambda)\Omega_+ = 0 \quad \text{and} \quad h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+,$$

but now we have

$$\rho(\lambda) = \sum_{m=1}^{N} \frac{s_m}{\sinh(\lambda + \alpha_m)\sinh(\lambda - \alpha_m)}.$$

ABA – trigonometric $s\ell(2)$ Gaudin with boundary

Again the vector Ω_+ is an eigenvector of $\tau(\lambda)$

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+$$
$$= 2\sinh^2(2\lambda)\left(\rho^2(\lambda) + \frac{\rho(\lambda)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)} - \frac{\rho'(\lambda)}{\sinh(2\lambda)}\right)\Omega_+.$$

The Bethe vectors

In this case, probably the simplest way defined the Bethe vectors is to consider the family of operators

$$\mathcal{C}_K(\mu) = f(\mu) + \frac{\psi}{\kappa}\left((2K-1) + \left(e^{-2\xi} - \cosh(2\mu)\right) h(\mu)\right) + \frac{\psi^2}{\kappa^2}\frac{e^{-\xi}}{2\sinh(\xi)}\left(e^{-2\xi} + 1 - 2\cosh(2\mu)\right) e(\mu),$$

for any natural number K.

The Bethe vectors

Then the Bethe vectors are given by

$$\varphi_1(\mu) = \mathcal{C}_1(\mu)\Omega_+, \quad \varphi_2(\mu_1,\mu_2) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\Omega_+$$

$$\varphi_3(\mu_1,\mu_2,\mu_3) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\mathcal{C}_3(\mu_3)\Omega_+.$$

Although in general the operators $\mathcal{C}_K(\mu)$ do not commute, it is effortless to confirm that the Bethe vector $\varphi_2(\mu_1,\mu_2)$ is a symmetric function

$$\varphi_2(\mu_1,\mu_2) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\Omega_+ = \mathcal{C}_1(\mu_2)\mathcal{C}_2(\mu_1)\Omega_+ = \varphi_2(\mu_2,\mu_1).$$

The Bethe vectors

Then the Bethe vectors are given by

$$\varphi_1(\mu) = \mathcal{C}_1(\mu)\Omega_+, \quad \varphi_2(\mu_1, \mu_2) = \mathcal{C}_1(\mu_1)\mathcal{C}_2(\mu_2)\Omega_+$$

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The Bethe vectors

Direct calculation shows that the off-shell action of the generating function $\tau(\lambda)$ on $\varphi_2(\mu_1,\mu_2)$ is given by

$$\tau(\lambda)\varphi_2(\mu_1,\mu_2) = \chi_2(\lambda,\mu_1,\mu_2)\varphi_2(\mu_1,\mu_2) + \sum_{i=1}^{2} \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)} \frac{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)} \times$$
$$\times \left(2\rho(\mu_i) + \frac{1}{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)} - \frac{2}{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i})}\right)\varphi_2(\lambda,\mu_{3-i}),$$

with the eigenvalue

$$\chi_2(\lambda,\mu_1,\mu_2) = \chi_0(\lambda) - \sum_{i=1}^{2} \frac{2\sinh^2(2\lambda)}{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i)} \times$$
$$\times \left(2\rho(\lambda) + \frac{1}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)} - \frac{1}{\sinh(\lambda+\mu_{3-i})\sinh(\lambda-\mu_{3-i})}\right).$$

The Bethe vectors

The two unwanted terms in the action above vanish when the Bethe equations are imposed on the parameters μ_1 and μ_2,

$$2\rho(\mu_i)+\frac{1}{\sinh(\xi+\mu_i)\sinh(\xi-\mu_i)}-\frac{2}{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i})}=0,$$

with $i=1,2$.

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- Analogously with the Gaudin model.
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- Outlook
  - We will further explore generalized Gaudin algebras by constructing new examples using the so-called twisting and fusion.
  - For the moment elliptic case is challenging due to some initial technical difficulties related to the representation theory of the so-called Sklyanin algebra.

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