Canonical Loop Quantum Gravity

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NARODOWE CENTRUM NAUKI

INTRODUCTION:

Canonical gravity

3-dimensional manifold: Σ

3-metric and momentum: $q_{ab}, \quad p^{ab}, \quad a, b = 1, 2, 3$

The Poisson bracket: $\{q_{ab}(x), p^{cd}(y)\} \; = \; \delta^c_{(a}\delta^d_{b)}\delta(x, y)$

The constraints:
$$C := \frac{1}{\sqrt{q}}\left(p^{ab}p_{ab} - \frac{1}{2}(p^a{}_a)^2\right) - \sqrt{q}R = 0$$

$$C_a := -2q_{ac}D_b p^{bc} = 0$$

Hamiltonian:
$$H = \int_\Sigma d^3x \, (NC + N^a C_a)$$

Canonical gravity

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The constraints: $$C^{\rm gr} := \frac{1}{\sqrt{q}}\left(p^{ab}p_{ab} - \frac{1}{2}p^2\right) - \sqrt{q}R$$

$$C^{\rm gr}_a := -2q_{ac}D_b p^{bc}$$

Canonical gravity and matter

$$C = C^{\mathrm{gr}} + H^{(\mathrm{matter})} \qquad \{\varphi(x), \pi(x')\} = \delta(x, x')$$

$$C_a = C_a^{\mathrm{gr}} + P_a^{(\mathrm{matter})}$$

$$C = 0 \qquad C_a = 0$$

$$G_i = 0$$

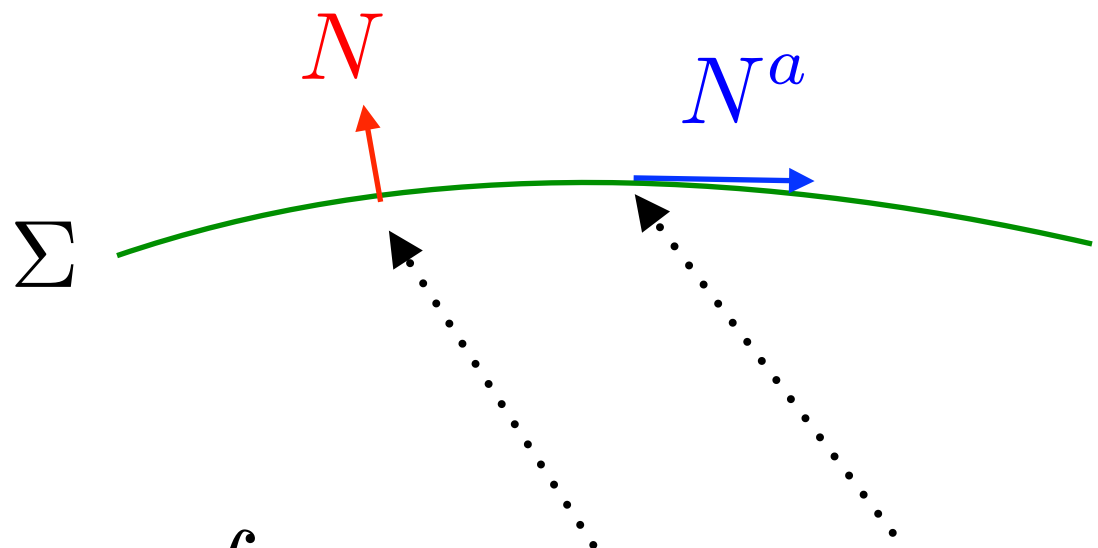
The Gauss constraint of the gauge fields

Still:

$$\boxed{H = \int_\Sigma (NC + N^a C_a + \Lambda^i G_i)}$$

The issue of the dynamics

Hamiltonian:



$$H = \int_\Sigma d^3x \, (NC + N^a C_a)$$

Physical observables:

$$\{\mathcal{O}, C\} = 0 = \{\mathcal{O}, C_a\}$$

$$\mathcal{O} = \mathcal{O}(q, p, \varphi, \pi)$$

$$\Downarrow$$

$$\frac{d\mathcal{O}}{dt} = \{\mathcal{O}, H\} = 0$$

Solution to the dynamics issue: the relational observables

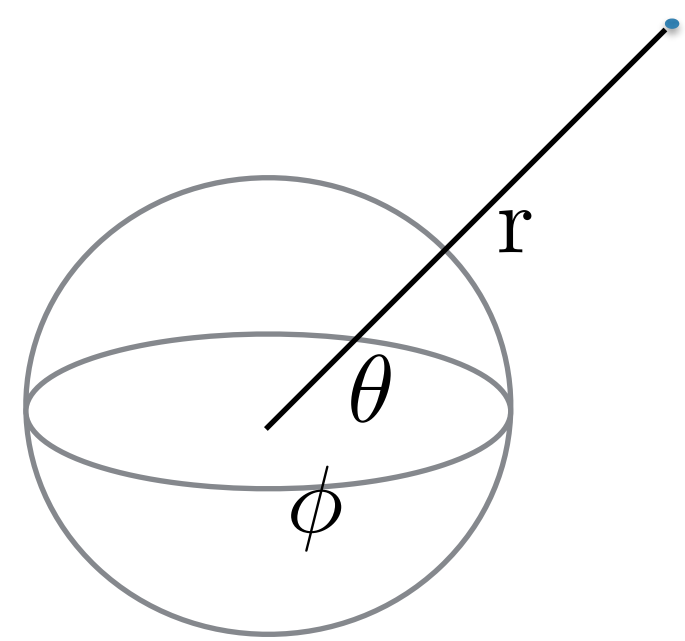
1. J. Kijowski (1990) - deparametrization of GR
2. C. Rovelli (1991) - initial values as observables
3. B. Dittrich (2006) - systematization
4. T. Thiemann (2006) - the book

5. A. Dapor, W. Kamiński, J. Lewandowski, and J. Świeżewski, (2013) - the subject revisited, several wrong statements pointed out and corrected

6. Bodendorfer, Duch, Lewandowski, Świeżewski (2016) - new idea, geometric construction of Dirac observables, example: a Gauss observer

The Gaussian gauge

Radial coordinates in Σ



Impose gauge conditions:

$$q_{rr} = 1, \quad q_{r\theta} = q_{r\phi} = 0$$

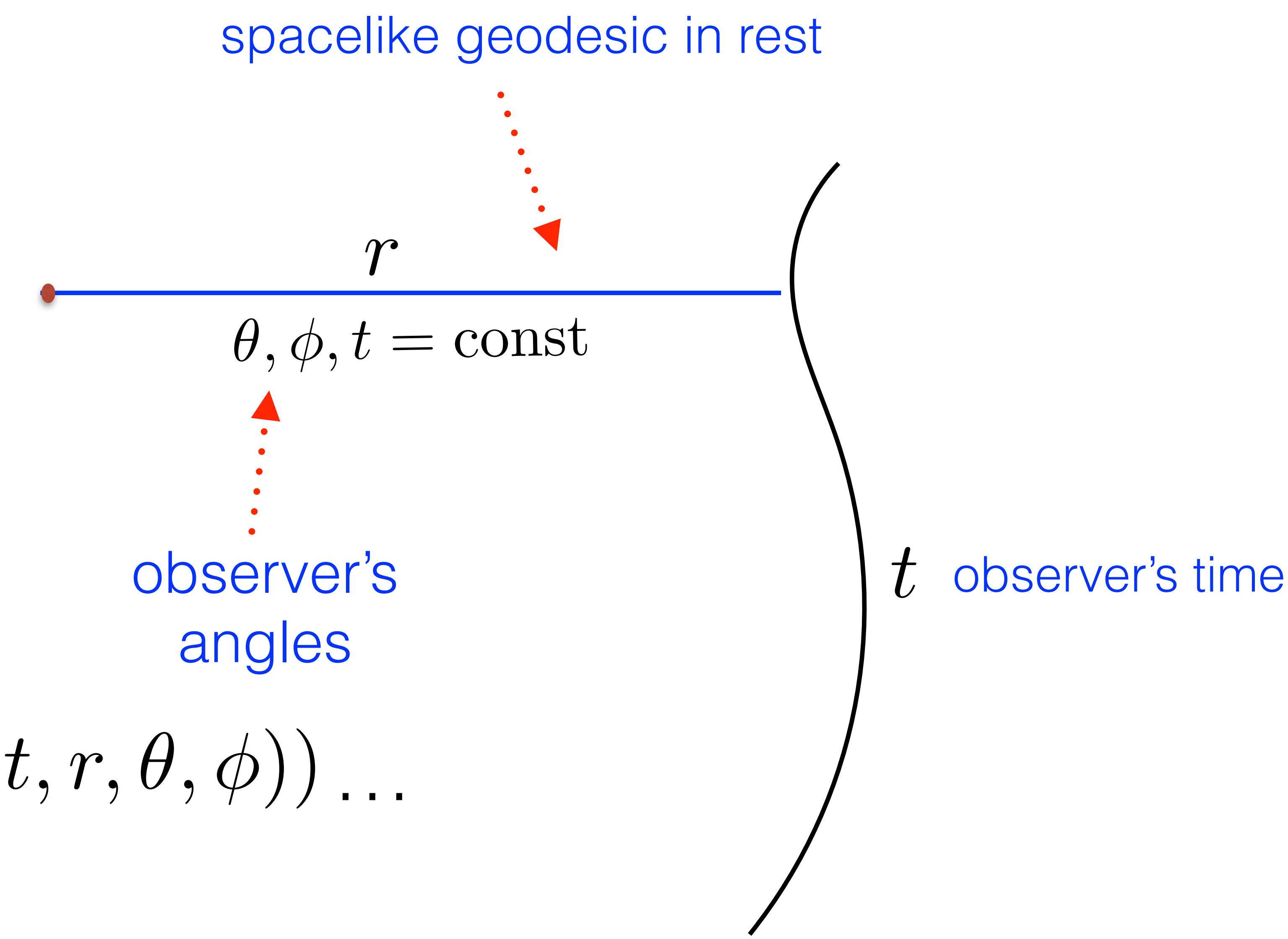
$$p^{rr} - \frac{1}{2} p^a_a = 0$$

Bodendorfer, Duch, Lewandowski, Świeżewski 2016

The Gaussian observer in spacetime

observers coordinates

$$(t, r, \theta, \phi)$$



Observables:

$$\mathcal{O} = \varphi(x(t, r, \theta, \phi))\,, \quad g_{\theta\phi}(x(t, r, \theta, \phi))\ldots$$

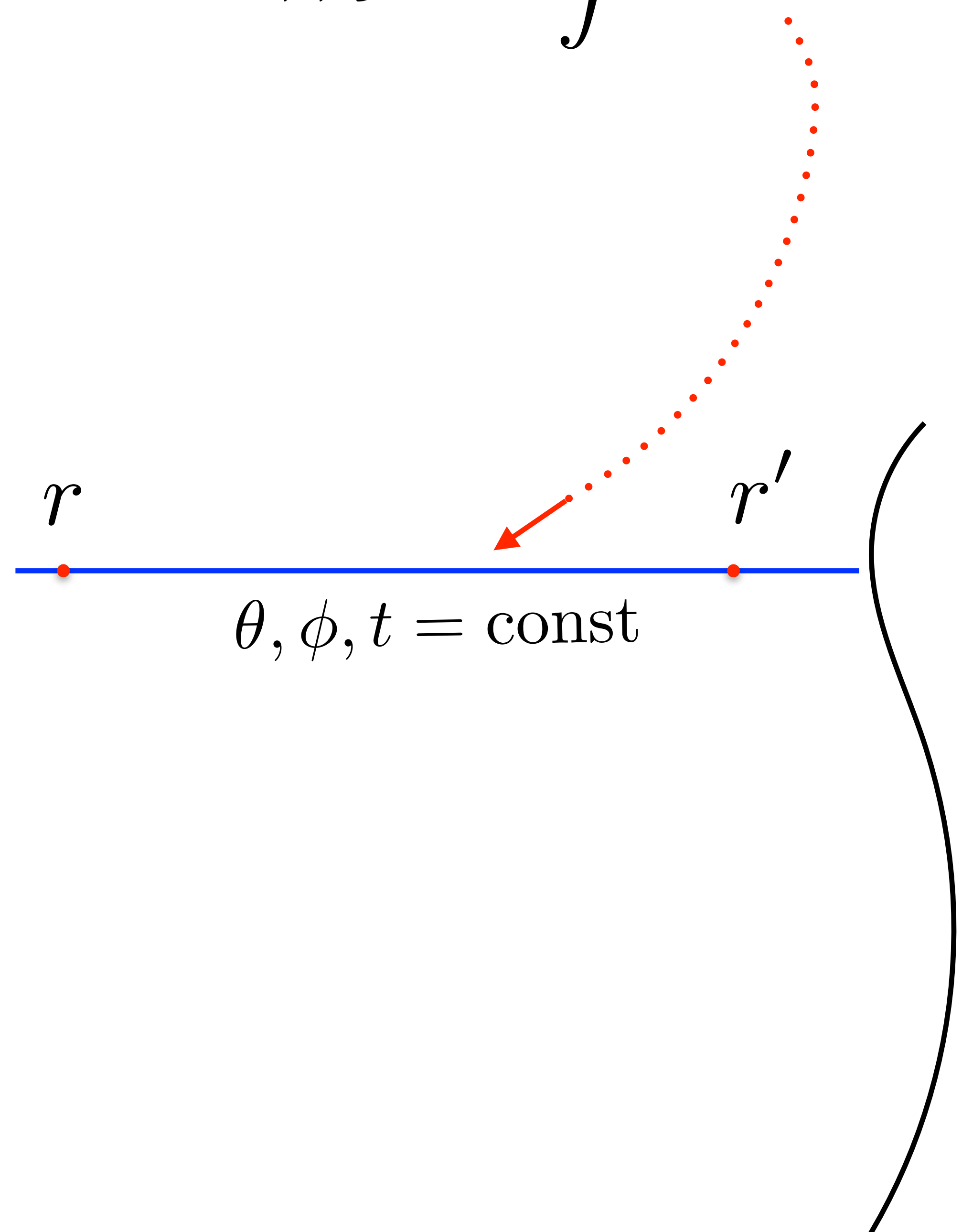
Dynamics:

$$\frac{\partial \mathcal{O}}{\partial t}$$

Observer's world line in the spacetime

Non-commuting of the corresponding observables

$$\{\varphi(x(t,r,\theta,\phi)),\varphi(x(t,r',\theta,\phi))\} \;=\int dr'' \ldots$$



Gaussian observer's symmetries: deformed Poincare

Every translation and infinitesimal Lorentz rotation

$$t \in T_{p_0}M, \qquad l \in so(g(p_0))$$

Defines an infinitesimal diffeomorphism of Σ

$$X^{(t,l)}$$

preserving the observer's coordinate system.

$$\begin{aligned} [\tilde{X}^{(0,l)}, \tilde{X}^{(0,l')}] &= \tilde{X}^{(0,[l',l])} \\ [\tilde{X}^{(t,0)}, \tilde{X}^{(t',0)}] &= \tilde{X}^{(0,l")} \end{aligned}$$

$$l"^{\mu}_{\ \nu} = t^{\alpha} t'^{\beta} R_{\alpha\beta\nu}{}^{\mu}(p_0)$$

Deformation

Deparametrization by a massless KG-scalar field

$$C = \frac{\pi^2}{2q} + \frac{1}{2\sqrt{q}} q^{ab}\varphi_{,a}\varphi_{,b} + C^{\rm gr} = 0$$

$$C_a = \pi\varphi_{,a} + C^{\rm gr}(q,p) = 0$$

$$\Rightarrow$$

$$\pi - h(q,p) = 0$$

$$\{h(x), h(y)\} = 0$$

Physical states

$$\left(\frac{\hbar}{i}\frac{\delta}{\delta\varphi} - \widehat{h(q,p)}\right)\Psi = 0$$

$$\hat{C}_a\Psi = 0$$

$$\Rightarrow$$

$$\Psi(q,\varphi) = e^{\frac{i}{\hbar}\int d^3x \hat{h}\varphi}\psi(q)$$

$$\psi(f^*q) = \psi(q), \qquad f:\Sigma\to\Sigma$$

Physical observables

$$[\widehat{\mathcal{O}}, \frac{\hbar}{i}\frac{\delta}{\delta\phi} - \widehat{h(q,p)}] = 0$$

$$[\widehat{\mathcal{O}}, \hat{C}_a] = 0$$

$$\Rightarrow$$

$$\hat{\mathcal{O}} = e^{\frac{i}{\hbar}\int d^3x \hat{h}\hat{\varphi}}\widehat{o(q,p)}e^{-\frac{i}{\hbar}\int d^3x \hat{h}\hat{\varphi}}$$

$$o(f^*q, f^*p) = o(q,p)$$

Dynamics by deparametrization

$$\hat{\varphi} \mapsto \hat{\varphi} + t \qquad \widehat{\mathcal{O}} \mapsto \mathcal{O}(t) := e^{\frac{i}{\hbar}\int d^3x \hat{h}(\hat{\varphi}+t)} \widehat{o(q,p)} e^{-\frac{i}{\hbar}\int d^3x \hat{h}(\hat{\varphi}+t)}$$

$$< \mathcal{O}(t) > = \qquad < \psi | e^{\frac{i}{\hbar}\widehat{H(q,p)}t} \widehat{o(q,p)} e^{-\frac{i}{\hbar}\widehat{H(q,p)}t} | \psi >$$

$$H = \int \hat{h}\, d^3x$$

In summary:
physical states are diffeomorphism invariant

$$\psi(q)$$

The dynamics is defined by $\varphi \mapsto \varphi + t$ and $\hat{H}$

Rovelli - Smolin (1993), Kuchar - Romano (1995), …, Domagala-Dziendzikowski-Lewandowski (2011)

Connection-frame variables

$$S(e,\omega) = \frac{1}{4\kappa}\int_M \epsilon_{IJKL} e^I \wedge e^J \wedge \Omega^{KL} - \frac{1}{2\kappa\beta}\int_M e^I \wedge e^I \wedge \Omega_{IJ}$$

Palatini

$$I, J, ... = 0, ..., 3$$

$$\omega_{IJ} = -\omega_{JI}$$

$$d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J \ = \ \Omega^I{}_J$$

1. General relativity in Ashtekar-Barbero variables
2. LQG framework
   - LQG Hilbert space & solutions of the kinematical constraints
   - Implementation of the scalar constraint
     - Non-symmetric constraint operator: Regularization
     - Adjoint operator & symmetric constraint operator
3. Summary, applications

General relativity in Ashtekar-Barbero variables

Action of 3+1 gravity

$$S = \int_{\mathbb{R}} dt \int_{\Sigma} d^3x \; \left[\frac{1}{k\beta} \dot{A}_a^i \, E_i^a - \left(\Lambda^j \, G_j + N^a \, C_a + N \, C \right) \right]$$

Ashtekar-Barbero variables

$$\{ A_a^i(x), E_j^b(y) \} = k\beta \delta_j^i \delta_a^b \delta(x,y), \qquad k = 8\pi G$$

Constraints

$$G_j(x) = \frac{1}{k\beta} D_a E_j^a(x)$$ — Gauss constraints

$$C_a(x) = \frac{1}{k\beta} F_{ab}^i(x) E_i^b(x)$$ — Vector constraints

$$C(x) = \frac{1}{2k\beta^2} \left[\frac{\epsilon^{ij}{}_k F_{ab}^k E_i^a E_j^b}{\sqrt{\left|\det(E_i^a)\right|}}(x) \; + \; (1 - s\beta^2) \sqrt{\left|\det(E_i^a)\right|} R(x) \right]$$ — Scalar constraints

s: spacetime signature

[Barbero (1994)], [Assanioussi, JL, Mäkinen (2015)]

Constraints (smeared)

$$\mathcal{G}(\Lambda) = \int_\Sigma d^3x\, \Lambda^i(x)\mathcal{G}_i(x)\,, \quad \vec{C}(\vec{N}) = \int_\Sigma d^3x\, N^a(x)C_a(x)\,, \quad C(N) = \int_\Sigma d^3x\, N(x)C(x),$$

The Gauss constraints generate:

$$A' = g^{-1}Ag + g^{-1}dg, \quad E' = g^{-1}Eg, \qquad g \in C(\Sigma, \mathrm{SU}(2))$$

The vector constraints generate:

$$A' = \varphi^* A, \quad E' = \varphi_*^{-1} E, \qquad \varphi \in \mathrm{Diff}(\Sigma)$$

$$\begin{aligned}
\{\mathcal{G}(\Lambda), \mathcal{G}(\Lambda')\} &= \mathcal{G}([\Lambda, \Lambda']), & \{\vec{C}(\vec{M}), \vec{C}(\vec{N})\} &= \vec{C}(\mathcal{L}_{\vec{M}}\vec{N}),\\
\{\mathcal{G}(\Lambda), \vec{C}(\vec{N})\} &= -\mathcal{G}(\mathcal{L}_{\vec{N}}\Lambda), & \{\vec{C}(\vec{M}), C(N)\} &= C(\mathcal{L}_{\vec{M}}N)\,,\\
\{\mathcal{G}(\Lambda), C(N)\} &= 0, & \{C(M), C(N)\} &= \vec{C}(q^{ab}[NM_{,b} - MN_{,b}]).
\end{aligned}$$

General relativity in Ashtekar-Barbero variables

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&\{\mathcal{G}(\Lambda), \vec{C}(\vec{N})\} = -\mathcal{G}(\mathcal{L}_{\vec{N}}\Lambda), && \{\vec{C}(\vec{M}), C(N)\} = C(\mathcal{L}_{\vec{M}}N)\ ,\\
&\{\mathcal{G}(\Lambda), C(N)\} = 0, && \{C(M), C(N)\} = \vec{C}(q^{ab}[NM_{,b} - MN_{,b}]).
\end{aligned}$$

Holonomy-flux algebra

$$h_e[A] \ = \ \mathcal{P}\exp\left(-\int_e A_a^i \tau_i\, dx^a\right) \quad , \quad P_{S,\xi} \ := \ \frac{1}{2}\int_S \epsilon_{abc}\xi^i(x)E_i^a(x)dx^b \wedge dx^c\ ;$$

The functions

$$\Psi(A) = \psi(h_{e_1}[A], ..., h_{e_n}[A])\ ,$$

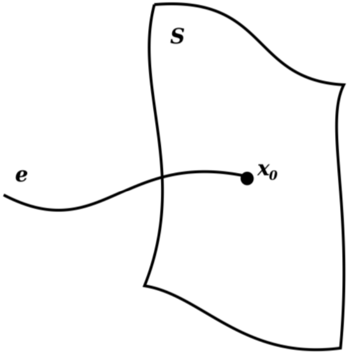
where $\{e_1, ..., e_n\} =: \Gamma$ are all embedded graphs in Σ, form the algebra Cyl.

Holonomy-flux algebra

$$\{D(h_e(A)), P_{S,\xi}(E)\} \;=\; -\frac{1}{2}D(h_e(A))D'(\xi(x_0))$$

The fluxes become derivations

$$P_{S,x} : \mathrm{Cyl} \to \mathrm{Cyl}$$



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Hilbert space

Kinematical Hilbert space

Our $\mathrm{Diff}(\Sigma)$ invariant integral defined on Cyl

$$\int \Psi(A) DA \ := \ \int dg_1 \dots dg_n \psi(g_1, \dots, g_n),$$

gives rise to the kinematical Hilbert space $\mathscr{H}_{\text{kin}}$:

$$\mathscr{H}_{\text{kin}} := \overline{\text{Cyl}} = \overline{\bigcup_{\Gamma} \text{Cyl}_{\Gamma}} = \bigoplus_{\Gamma} \mathcal{H}_{\Gamma}$$

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Hilbert space of $SU(2)$-gauge invariant states

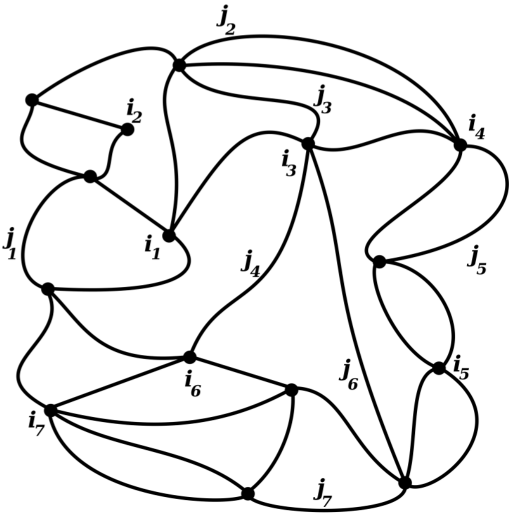
The space of solutions of the quantum Gauss constraint operator is the subspace $\mathscr{H}^{G}_{\text{kin}} \subset \mathscr{H}_{\text{kin}}$ of gauge invariant functions. Can be obtained by the averaging:

$$(\eta^{\text{Gauss}} \Psi)(A) := \int dg_1 ... dg_k \psi(g^{-1}_{j_1} h_{e_1}(A) g_{i_1}, ..., g^{-1}_{j_n} h_{e_n}(A) g_{i_n})$$

$$\mathscr{H}^{G}_{\text{kin}} = \bigoplus_{\Gamma} \mathscr{H}^{G}_{\Gamma}$$

[Ashtekar, JL (1993)]

Hilbert space



$j_1, j_2, j_3, \ldots \in \frac{1}{2}\mathbb{N}$, $\iota \in \mathrm{Inv} \bigotimes_i V_{j_i} \otimes \bigotimes_l V^*_{j_l}$ [Rovelli, Smolin (1995)], [Baez (1995)]

Mathematical structures

Cyl with the sup-norm defines an Abelian C^*-algebra. The integral $\int$ DA defines a Diff(Σ) invariant measure on the Gel'fand spectrum.
[Ashtekar, JL (1993)], [Marolf, Mourao (1994)], [Baez, Sawin (1995)]

The fluxes and Cyl define a quantum *-algebra $\mathcal{A}$ obtained by replacing

$$\{\cdot,\cdot\} \mapsto \frac{1}{i\hbar}[\cdot,\cdot]$$

The algebra $\mathcal{A}$ admites a unique diffeomorphism invariant state

$$\omega : \mathcal{A} \to \mathbb{C}$$

Our kinematical quantization is equivalent to the GNS with that state.
[JL, Okolow, Sahlmann, Thiemann (2005)]

The spatial diffeomorphism invariant states

Hilbert space of spatial diffeomorphism invariant states $\mathscr{H}^G_{\rm Diff}$

The space of solutions to the vector constraints is constructed through group averaging using a rigging map η,

$$\begin{aligned} \eta \ : \ \mathscr{H}^G_\Gamma &\mapsto \mathrm{Cyl}^* \\ \psi_\Gamma &\mapsto \frac{1}{n_\Gamma} \sum_{[\varphi]\in \mathrm{Diff}/\mathrm{TDiff}_\Gamma} \langle U(\varphi)\psi_\Gamma| =: \eta(\psi_\Gamma) \end{aligned}$$

n_Γ - averaging coefficient.

Then the space of the Gauss and spatial diffeomorphism invariant states is defined as

$$\mathscr{H}^G_{\rm Diff} := \overline{\eta\left(\mathscr{H}^G_{\rm kin}\right)} \subset \mathrm{Cyl}^*$$

[Ashtekar, JL, Marolf, Mourao, Thiemann (1995)]

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[Ashtekar, JL, Marolf, Mourao, Thiemann (1995)]

- $\mathscr{H}^G_{\text{Diff}}$ is *not preserved* by any operator of the form $\hat{\mathscr{O}}(N)$, because of the lapse function N.
- In case of a scalar constraint operator, this is a serious issue in the treatment of several questions such as self-adjointness and spectral resolution.

Solution: PRD 91, 044022 (2015) [Arxiv: 1410.5276], JL, Sahlmann

- Construct a (dual) space of *"partially"* diff. invariant states that is preserved by such scalar constraint operator.

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Hilbert space of *partially* Diff. invariant states $\mathscr{H}_{\text{vtx}}^{G}$

Average only w.r.t. diff. that act trivially on the vertices of the graph of a given state,

$$\psi_\Gamma \mapsto \frac{1}{n_\Gamma} \sum_{[\varphi]\in \mathrm{Diff}_{\mathrm{Vtx}(\Gamma)}/\mathrm{TDiff}_\Gamma} \langle U(\varphi)\psi_\Gamma| =: \eta(\psi_\Gamma).$$

The space of the Gauss and partially diff. invariant states is defined as $\mathscr{H}_{\text{vtx}}^{G} := \overline{\eta\left(\mathscr{H}_{\text{kin}}^{G}\right)}$,

$$\mathcal{H}_{\text{vtx}}^{G} = \bigoplus_X \mathcal{H}_X^G, \qquad X \subset \Sigma, \quad |X| < \infty\,.$$

The resulting Hilbert space is preserved by every quantum scalar constraint operator

$$\hat{C}(N) : \mathscr{H}_{\text{vtx}} \to \mathscr{H}_{\text{vtx}}$$

The Hilbert space $\mathscr{H}^G_{\text{vtx}}$

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The resulting Hilbert space is preserved by every quantum scalar constraint operator

$$\hat{C}(N) : \mathscr{H}_{\text{vtx}} \to \mathscr{H}_{\text{vtx}}$$

- The treatment and discussion of the properties of the constraint operator is generalized from $\mathscr{H}^G_{\text{Diff}}$ to $\mathscr{H}^G_{\text{vtx}}$;
- $\hat{C}^\dagger(N)$ is densely defined in $\mathscr{H}^G_{\text{vtx}}$ $\rightarrow$; that allows symmetrization.

Regularization of the scalar constraint

$$C(N) = \frac{1}{2k\beta^2} \int_\Sigma d^3x \, N(x) \left(\frac{\epsilon_{ijk} E^a_i(x) E^b_j(x) F^k_{ab}(x)}{\sqrt{|\det E(x)|}} + \left(1 - s\beta^2\right) \sqrt{|\det E(x)|} \, R(x) \right),$$

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Euclidean part

$$C^E(N)=\int_\Sigma d^3x\,N(x)\,\frac{\epsilon_{ijk}E^a_i(x)E^b_j(x)F^k_{ab}(x)}{\sqrt{|\det E(x)|}}$$

- Thiemann's shuffle

  $$\mathrm{sgn}(\det(e))\,\frac{\epsilon_{ijk}E^a_i(x)E^b_j(x)}{\sqrt{|\det E(x)|}}=\frac{2}{k}\epsilon^{abc}\{A^k_c(x),V\}\,;$$

- $F_{ab}\longrightarrow h_{\alpha_{ab}}$;
- The loop α_{ab} does NOT overlap with the graph of the state in the regularization;
- Tangentiality conditions for the assignment of a loop at a given node.

Regularization of the scalar constraint

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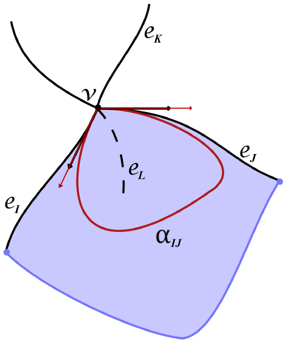
Lorentzian part

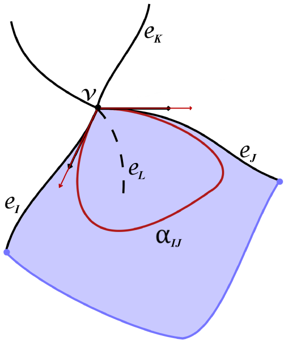
$$C^L(N) = \int_\Sigma d^3x\, N(x) \sqrt{|\det E(x)|}\, R(x)$$

- Regge's approximation
- External regularization;
- $\longrightarrow$ The curvature operator;
  [E.A., M.A., J.L. PRD 89, 124017 (2014), arXiv:1403.3190]

The Euclidean operator

- choice of a specific coordinate plane (adapted frame) with a proper routing;
- imposition of tangentiality conditions;





- choice of a specific coordinate plane (adapted frame) with a proper routing;
- imposition of tangentiality conditions;

With this prescription

- the loop assigned to a pair of edges is unique up to diffeomorphisms;
- This prescription makes a loop assigned to a given pair of edges perfectly distinguishable from any other loop at the same node;

$$\hat{C}^E_\epsilon(N)\Psi_\gamma \propto \sum_{v\in\gamma}\sum_{I,J,K}\frac{N(v)}{E_v}\epsilon^{IJK}\mathrm{Tr}\Big(h^{(l)}_{\alpha_{IJ}(\Delta)}h^{(l)}_{e_K(\Delta)}[h^{(l)\,-1}_{e_K(\Delta)},\hat{V}]\Big)\Psi_\gamma$$

E_v - averaging coefficient $\qquad$ l - arbitrary spin representation.

Non symmetric scalar constraint operator

Euclidean part

$$\hat{C}^E(N) := \lim_{\epsilon\to 0}\left[\hat{C}^E_\epsilon(N)\right]^*$$

- Gauge invariant and diff. covariant;
- Graph changing without creating vertices;
- Preserves $\mathscr{H}^G_{\text{vtx}}$;
- Densily defined on $\mathscr{H}^G_{\text{vtx}}$.

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Lorentzian part

$$\hat{C}^L(N)\Psi_\gamma \propto \sum_{\substack{v \in \gamma \\ e_I \cap e_J = v}} N(v)\kappa_v \left[\underbrace{\hat{L}_{IJ}(\hat{E}_i)}_{\substack{\text{Length operator} \\ \text{contains } \widehat{V^{-1}}}} \underbrace{\hat{\Theta}_{IJ}(\hat{E}_i)}_{\text{Angle operator}} \right]^* \Psi_\gamma$$

- Gauge invariant and diff. covariant;
- Non-graph changing;
- Preserves $\mathscr{H}^G_{\text{vtx}}$;
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κ_v - averaging coefficient

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- Densily defined on $\mathscr{H}^G_{\text{vtx}}$.

Lorentzian part

$$\hat{C}^L(N)\Psi_\gamma \propto \sum_{\substack{v\in\gamma \\ e_I\cap e_J=v}} N(v)\kappa_v \Big[\underbrace{\hat{L}_{IJ}(\hat{E}_i)}_{\substack{\text{Length operator} \\ \text{contains } \widehat{V^{-1}}}}\ \underbrace{\hat{\Theta}_{IJ}(\hat{E}_i)}_{\text{Angle operator}}\Big]^*\Psi_\gamma$$

- Gauge invariant and diff. covariant;
- Non-graph changing;
- Preserves $\mathscr{H}^G_{\text{vtx}}$;
- Densily defined on $\mathscr{H}^G_{\text{vtx}}$.

κ_v - averaging coefficient

$$\hat{C}(N) := \hat{C}^E(N) + (1 - s\beta^2)\hat{C}^L(N)$$

- The scalar constraint operator does not create new vertices but new links;
- Has a similar action to the scalar constraints in the symmetry reduced cosmological models
- The treatment and discussion of the properties of the constraint operator can be performed directly in $\mathscr{H}^G_{\text{vtx}}$;

Symmetric constraint operator

Adjoint operator of $\hat{C}(N)$

At this level we define the operator $\hat{C}(N)$ on a dense domain $\mathscr{D}[\hat{C}(N)]$ in $\mathscr{H}^{G}_{\mathrm{vtx}}$.

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Symmetric extensions

$$\hat{C}_{\text{Sym}}(N) := \text{Sym}(\hat{C}(N), \hat{C}^\dagger(N))$$

Typical example: $\frac{1}{2}(\hat{C}(N) + \hat{C}^\dagger(N))$

Summary

We have introduced the Hilbert space

$$\mathcal{H}^G_{\text{vtx}} = \bigoplus_{\{v_1,\dots,v_k\}} \mathcal{H}_{\{v_1,\dots,v_k\}}$$

which admits quantum volume element, quantum Ricci scalar, quantum scalar constraint

$$\int_\Sigma d^3x\sqrt{\hat{q}(x)}N(x),\ \int_\Sigma d^3x\sqrt{\hat{q}(x)}\hat{R}(x)N(x),\ \int_\Sigma d^3x\hat{C}(x)N(x),$$

as well as physical observables containing $\hat{F}^i_{ab}$.
Solutions to the vacuum constraint

$$< \Psi|\hat{C}(N) = 0$$

are defined via the spectral decomposition

$$\mathcal{H}^G_{\text{vtx}} = \bigoplus_c \mathcal{H}_c$$

Applications

Gravity deparametrized by a coupled massless scalar field or non-rotating dust amounts to a quantum theory defined in $\mathcal{H}^{G}_{\text{vtx}}$ by the equation

$$i\frac{d}{dt}\Psi \ = \ H\Psi,$$

with the Hamiltonian being

$$H = \int d^3x \sqrt{-2\sqrt{\hat{q}}\,\hat{C}}$$

or

$$H = -\int d^3x\,\hat{C},$$

respectively.

[Domagała, Dziendzikowski, JL (2012)], [Assanioussi, JL, Mäkinen (2016)]

Perturbative dynamics in deparametrized models

Quantum evolution:

$$\hat{U}(t) := exp[-it\hat{H}] \qquad , \qquad \hat{H} = f(\beta)\left(\hat{C}^L + \frac{1}{1+\beta^2}\hat{C}^E\right)$$

- Transition amplitudes: $\mathscr{A}_{ij}(t) = \langle\Psi_j|\,\hat{U}(t)\,|\Psi_i\rangle$
- Quantum observables: $\langle\mathscr{O}(t)\rangle$
- Perturbation theory for the dynamics:

$$\hat{H}_0 := \hat{C}^L \qquad , \qquad \hat{V} := \hat{C}^E \qquad , \qquad \epsilon := -\frac{1}{1+\beta^2}$$

$$\hat{H} = f(\beta)(\hat{H}_0 + \epsilon\hat{V}) \qquad , \qquad |\epsilon| \ll 1 \Leftrightarrow \beta^2 \gg 1$$

$$\longrightarrow \mathscr{A}_{ij}(t) = \mathscr{A}_{ij}^{(0)}(t) + \epsilon\,\mathscr{A}_{ij}^{(1)}(t) + \epsilon^2\,\mathscr{A}_{ij}^{(2)}(t) + \ldots$$

[M. Assanioussi, J. Lewandowski and I. Mäkinen (2017)]

Rainbow gravity

Quantization of matter and gravity: Schroedinger-like equation

$$-i\hbar\frac{d}{dt}\Psi = \left[\hat{H}_o - \frac{1}{2}\left(\hat{H}_o^{-1}\otimes\hat{\pi}_k^2 + \hat{\Omega}(k,m)\otimes\hat{\phi}_k^2\right)\right]\Psi$$

$\longrightarrow \Psi = \Psi_o \otimes \varphi$, where $\varphi \in L_2(\mathbb{R}, d\phi_k)$ and $\Psi_o \in \mathcal{H}_G$ evolves via Schroedinger-like equation $-id\Psi_o/dt = \hat{H}_o\Psi_o$. This being the case, we can trace away the gravitational part and obtain an equation for the matter part only:

$$i\hbar\frac{d}{dt}\varphi = \hat{H}_k^{\text{fun}}\varphi\,, \qquad \hat{H}_k^{\text{fun}} := \frac{1}{2}\left[\langle\Psi_o|\hat{H}_o^{-1}|\Psi_o\rangle\hat{\pi}_k^2 + \langle\Psi_o|\hat{\Omega}(k,m)|\Psi_o\rangle\hat{\phi}_k^2\right]$$

On the other hand, constructing regular QFT on such a Robertson-Walker type spacetime, one obtains for mode k of ϕ:

$$i\hbar\frac{d}{dt}\varphi = \hat{H}_{\vec{k},m}^{\text{eff}}\varphi\,, \qquad \hat{H}_{\vec{k},m}^{\text{eff}} := \frac{1}{2}\left[\frac{\bar{N}}{\bar{a}^3}\hat{\pi}_k^2 + \frac{\bar{N}}{\bar{a}^3}(k^2\bar{a}^4 + m^2\bar{a}^6)\hat{\phi}_k^2\right]$$

In other words, we can replace the fundamental theory described with regular QFT on curved spacetime, provided that the terms in the two Hamiltonians, fundamental and effective, match. *[M. Assanioussi, A. Dapor and J. Lewandowski (2015)]*

Thank you

The scalar field deparamtrizes a space dimension rather than time

$$\int d^3x N(x)\sqrt{\hat{\varphi}_a(x)\hat{\varphi}_b(x)\hat{E}_i^a(x)\hat{E}_i^b(x)}\;\Psi_\Gamma \otimes|\varphi > \quad =$$

$$8\pi\beta\ell_{\mathrm{pl}}\,(\sum_e \sqrt{j_e(j_e+1)}\int_e N|d\varphi|)\quad \Psi_\Gamma \otimes|\varphi >$$

e - runs through the set of the edges (links) of the spin-network Γ