Arithmetic quantum chaos and random wave conjecture

9th Mathematical Physics Meeting

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Abstract

A fundamental problem in the area of quantum chaos is to understand the distribution of large frequency eigenfunctions of the Laplacian on certain negatively curved Riemannian manifolds. Arithmetic quantum chaos refers to quantum systems that have arithmetic structure and so, are of interest to both number theorists and mathematical physicists. Such examples arise as hyperbolic surfaces obtained as quotients of the upper half-plane by a discrete arithmetic subgroup of $SL_2(\mathbb{R})$. The random wave conjecture says that an eigenform of large Laplacian eigenvalue (which is also a joint eigenform of all Hecke operators) should behave like a random wave, that is, its distribution should be Gaussian. In this talk in particular we focus on this conjecture in the case of Eisenstein series. This is based on the joint work with Rizwanur Khan.

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Classical mechanics: a point particle moving without friction in a billiard table Ω

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$\phi_j(x)$ - the amplitude of a stationary solution of Schrödinger's equation
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$$\int_\Omega |\phi_j|^2 d\mu = 1, \qquad \text{normalized, of unit } L^2\text{-norm}$$

Basic questions:

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots$$

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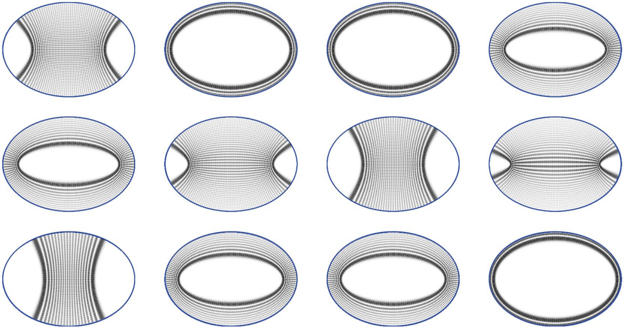
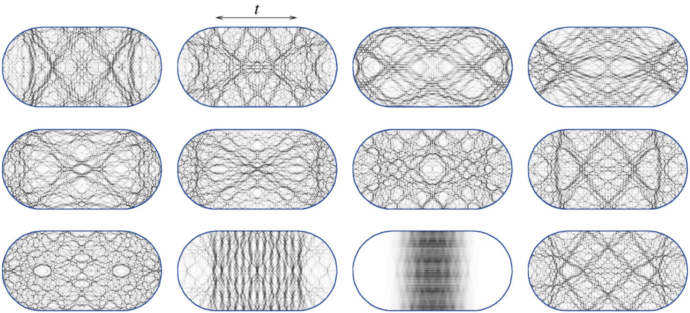
- How is the classical mechanics description reflected in the quantum description when Planck's constant $\hbar$ is small (or equivalently in the case at hand, when $\lambda_j \to \infty$)?
- Are there universal laws in the energy spectrum?
- What are the statistical properties of highly excited eigenfunctions?



Ω - an ellipse, 12 modes around 5600th eigenvalue,

classical Hamiltonian dynamics – a billiard in Ω

-- motion is integrable



Ω - a stadium, 12 modes around 5600th eigenvalue,

classical Hamiltonian dynamics – a billiard in Ω

-- motion is ergodic (almost all of the trajectories are dense)

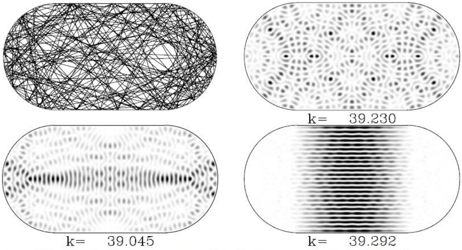
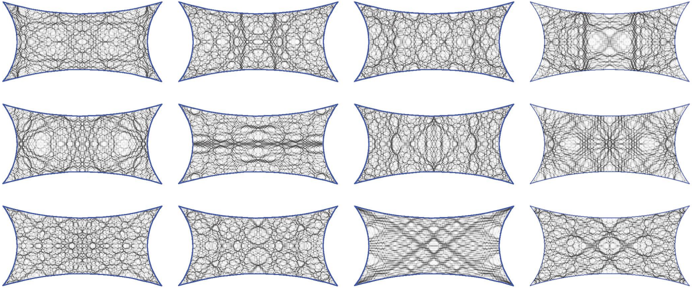


Figure 2. Top left: one typical "ergodic" orbit of the "stadium": it equidistributes across the whole billiard. The three other plots feature eigenmodes of frequencies $k_n \approx 39$. Bottom left: a "scar" on the (unstable) horizontal periodic orbit. Bottom right: a "bouncing ball" mode.



Ω - a dispersing Sinai billiard, 12 modes around 5600th eigenvalue,

classical Hamiltonian dynamics – a billiard in Ω

-- motion is ergodic and strongly chaotic (almost all of the trajectories are dense)

Density measures on Ω

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In the case of an ellipse (more generally - the quantization of any completely integrable Hamiltonian system)
$\to$ these measures (or rather their microlocal lifts μ_{ϕ_j} to $T_1(\Omega)$...) localize on invariant tori in a well understood manner

Compact manifolds

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the classical mechanics is that of motion by geodesics on the unit tangent bundle $T_1(M)$

The case of ergodic geodesic flow

ergodic – the only flow invariant subsets of $T_1(M)$ are either of zero or full μ-measure, where μ is the Liouville measure on $T_1(M)$

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Theorem (Shnirelman 1974, Zelditch 1987, Colin de Verdiere 1985)

If the geodesic flow is ergodic, then almost all (in the sense of density) of the eigenfunctions become equidistributed with respect to μ.
That is, if $\{\phi_j\}_{j=0}^{\infty}$ is an orthonormal basis of eigenfunctions with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$, then there is a subsequence j_k of integers of full density, such that

$$\mu_{j_k} \longrightarrow \mu, \qquad as \qquad k \to \infty$$

Quantum unique ergodicity conjecture

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★ Can the arclength measure on a closed geodesic ("strong scars", the most singular flow invariant measure) be a quantum limit?

Conjecture (QUE, Rudnick-Sarnak, 1994)

If M is a compact negatively curved manifold, then $\mu_\phi \to \mu$ as $\lambda \to \infty$, i.e. μ is the only quantum limit!

Stadium domain

Hassell (Annals of Math. 2010.) For almost all stadiums,

billiards are not quantum unique ergodic!

(there exist a quantum limit which gives positive mass to the *bouncing ball* trajectories)

Arithmetic QUE

$$\Gamma \leq PSL_2(\mathbb{R}), \qquad \text{a discrete subgroup}, \qquad \Gamma \circlearrowright \mathbb{H}$$

$\mathbb{H}$ – the hyperbolic plane

$$\Gamma = SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}$$

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$$M = \Gamma \backslash \mathbb{H}, \qquad \text{modular surface, noncompact, but of finite area}$$

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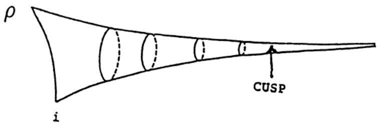
of constant curvature $K = -1$

– for these, there is also **continuous spectrum** coming from the theory of Eisenstein series developed by Selberg

Modular surfaces



$SL_2(Z)\backslash\boldsymbol{H}$

$\Gamma\backslash\boldsymbol{H}$

Hecke operators

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on $SL_2(\mathbb{Z})\backslash\mathbb{H}$ for every $n \geq 1$ we have the Hecke operator

$$T_n\phi(z) = \sum_{\substack{ad=n \\ 0\leq b<d}} \phi\left(\frac{az+b}{d}\right), \qquad T_n : L^2(M) \to L^2(M)$$

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$\{T_n\}_{n\geq 1}$ commuting family of normal operators which commute with hyperbolic Laplacian $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$
– can be simultaneously diagonalized ($\rightsquigarrow$ **Hecke eigenforms**)

Hecke-Maass forms on $M = SL_2(\mathbb{Z})\backslash\mathbb{H}$

- $\phi(\gamma z) = \phi(z)$, for all $\gamma \in SL_2(\mathbb{Z})$
- $\Delta\phi + \lambda\phi = 0$
- $\phi \in L^2(M)$
- $T_n\phi = \lambda_\phi(n)\phi$, for all $n \geq 1$

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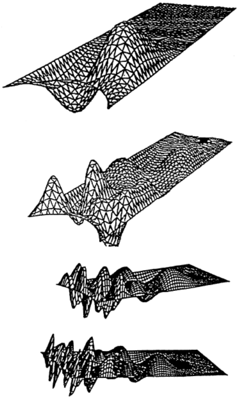
$\rightsquigarrow$ *hope*: QUE questions for arithmetic manifolds can be studied by methods from the theories of automorphic forms and associated L-functions

$1^{st}, 10^{th}, 17^{th}$ and 33^{rd} Hecke-Maass eigenfunction



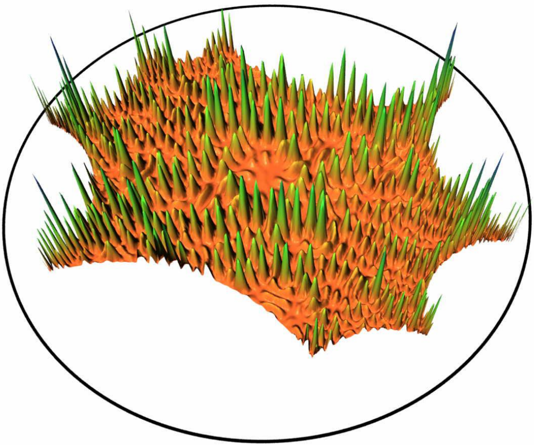
$\lambda_1 = 91.12\ldots$

$\lambda_{10} = 379.90\ldots$

$\lambda_{17} = 541.27\ldots$

$\lambda_{33} = 916.52\ldots$

One $|\phi|^2(z)$ for a non-arithmetic surface $\Gamma\backslash\mathbb{H}$ of genus two



Arithmetic QUE is true!

Theorem (E. Lindenstrauss, Ann. of Math. 2006)

Let M be a **compact** *arithmetic surface. Then the only quantum limit is the Liouville measure μ.*

proof: measure rigidity for higher rank diagonal actions on homogeneous spaces

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proof: measure rigidity for higher rank diagonal actions on homogeneous spaces

Theorem (E. Lindenstrauss, 2006 + K. Soundararajan, Ann. of Math. 2010)

Let M be a **noncompact** *arithmetic surface. Then QUE holds for both the continuous and discrete Hecke eigenfunctions on M.*

proof: + theory of multiplicative functions to eliminate an "escape of mass into cusp"

Random Wave Conjecture

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Conjecture (Random wave conjecture)

In the case of negative curvature, the Laplace eigenfunctions ϕ_λ *tend to exhibit Gaussian random behavior in the high energy limit.*

The moment version

Conjecture (RWC - the moment version)

For any integer $2 \leq p < \infty$ and any nice, compact $\Omega \subset SL_2(\mathbb{Z})\backslash\mathbb{H}$ we have

$$\frac{1}{\mathrm{vol}(\Omega)} \int_\Omega \phi_\lambda^p(z) \frac{dxdy}{y^2} \longrightarrow \sigma^p c_p, \qquad \lambda \to \infty,$$

where c_p is the pth moment of the normal distribution $\mathcal{N}(0;1)$ and $\sigma^2 = \frac{1}{\mathrm{vol}(SL_2(\mathbb{Z})\backslash\mathbb{H})} = \frac{3}{\pi}$ is the conjectured variance of the random wave.

Spectral decomposition of Δ on $SL_2(\mathbb{Z})\backslash\mathbb{H}$

$$f(z) = \sum_{j\geq 0}\langle f,\phi_j\rangle\phi_j(z) + \frac{1}{4\pi}\int_{-\infty}^{+\infty}\langle f, E(\cdot,\frac{1}{2}+it)\rangle E(z,\frac{1}{2}+it)dt$$

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where

- $\phi_0(z) = (3/\pi)^{1/2}$
- $\phi_j(z)$, $j \geq 1$ – cusp forms (discrete spectrum)
- $E(z,\frac{1}{2}+it)$ – Eisenstein series (continuous spectrum)
  – obtained as the analytic continuation (by Selberg) in s-variable of

$$E(z,s) := \sum_{\Gamma_\infty\backslash\Gamma}\Im(\gamma z)^s, \qquad \Re(s) > 1$$

Random Wave Conjecture for cusp forms

Theorem (J. Buttcane, R. Khan, 2016)

Assume the Generalized Lindelöf Hypothesis. Let f be an even or odd Hecke-Maass **cusp** *form for $M = SL_2(\mathbb{Z})\backslash\mathbb{H}$ with Laplacian eigenvalue $\lambda = \frac{1}{4} + T^2$, where $T > 0$. Let f be normalized to have probability measure equal to 1, as follows:*

$$\frac{1}{\mathrm{vol}(M)} \int_M |f(z)|^2 \frac{dxdy}{y^2} = 1.$$

There exists a constant $\delta > 0$ such that

$$\frac{1}{\mathrm{vol}(M)} \int_M |f(z)|^4 \frac{dxdy}{y^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^4 \, e^{-t^2/2} dt + O(T^{-\delta}), \qquad T \to \infty.$$

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– confirms the RWC, with a power saving, for cusp forms, conditionally on GLH

Random Wave Conjecture for Eisenstein series

Theorem (G. Dj., R. Khan, 2017)

Let $\{\phi_j : j \geq 1\}$ denote an orthonormal basis of even and odd Hecke-Maass cusp forms for $SL_2(\mathbb{Z})$, ordered by Laplacian eigenvalue $\frac{1}{4} + it_j^2$, and let $\Lambda(s, \phi_j)$ denote the corresponding completed L-functions. Let $\xi(s)$ denote the completed Riemann ζ function. As $T \to \infty$, we have

$$\int_M^{reg} |E(z, 1/2 + iT)|^4 \frac{dxdy}{y^2}$$

$$= \frac{24}{\pi} \log^2 T + \sum_{j \geq 1} \frac{\cosh(\pi t_j)}{2} \frac{|\Lambda(\frac{1}{2} + 2Ti, \phi_j)|^2 \Lambda^2(\frac{1}{2}, \phi_j)}{L(1, sym^2 \phi_j)\, |\xi(1 + 2Ti)|^4} + O(\log^{5/3+\epsilon} T),$$

for any $\epsilon > 0$.

RWC for the regularized 4^{th} moment

Conjecture (G. Dj., R. Khan, 2017 – RWC for the regularized fourth moment of Eisenstein series)

$$\int_M^{reg} |E(z, \frac{1}{2} + iT)|^4 \frac{dxdy}{y^2} \sim \frac{72}{\pi} \log^2 T.$$

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hope: to understand the sum of special values of L-functions in family of Hecke-Maass forms, by methods from *analytic number theory*

Triple product formula

T. Watson, 2001 – formula relating the integral of a product of three automorphic forms on an arithmetic surface to the special value of a degree 8 L-function

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– in the case of $M = SL_2(\mathbb{Z})\backslash\mathbb{H}$ and three Hecke-Maass cusp forms ϕ_1, ϕ_2, ϕ_3 whose L^2-norms are normalized to be 1 :

$$\left|\int_M \phi_1(z)\phi_2(z)\phi_3(z)\right|^2 = \frac{\pi^4}{216} \frac{\Lambda(1/2, \phi_1 \otimes \phi_2 \otimes \phi_3)}{\Lambda(1, \mathrm{sym}^2\phi_1)\Lambda(1, \mathrm{sym}^2\phi_2)\Lambda(1, \mathrm{sym}^2\phi_3)}$$

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$\Lambda(s, \phi_1 \otimes \phi_2 \otimes \phi_3)$ – L-function of degree 8
(Riemann zeta function $\zeta(s)$ is of degree 1)

Thank you!