Generalized plane waves in Poincaré gauge theory of gravity

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Talk outline

Talk is based on the paper:

- M. Blagojević, B. Cvetković and Y. N. Obukhov, Phys. Rev. D 96, 064031 (2017)

- The gauge principle, originally formulated by Weyl in the context of electrodynamics, now belongs to the key concepts which underlie the modern understanding of dynamical structure of fundamental physical interactions.
- Development of Weyl's idea, most notably in the works of Yang, Mills and Utiyama, resulted in the construction of the general gauge-theoretic framework for arbitrary non-Abelian groups of *internal* symmetries.
- Sciama and Kibble extended this formalism to the *spacetime* symmetries, and proposed a theory of gravity based on the Poincaré group.
- The importance of the Poincaré group in particle physics strongly supports the Poincaré gauge theory (PGT) as the most appropriate framework for description of the gravitational phenomena.

- "Translational" gauge field potentials can be identified with the spacetime coframe field and the "rotational" gauge field potentials can be interpreted as the spacetime connection.
- This introduces the Riemann–Cartan geometry on the spacetime manifold, since one recovers the *torsion* and the *curvature* as the Poincaré gauge field strengths.
- The gravitational dynamics in PGT is determined by a Lagrangian that is assumed to be the function of the field strengths and the dynamical setup is completed by including a suitable matter Lagrangian.
- In the past, investigations of PGT were mostly focused on the class of models with quadratic parity symmetric Lagrangians of the Yang-Mills type, expecting that the obtained results should be sufficient to reveal essential dynamical features of the more complex general theory.

- Recently, there has been a growing interest for the extended class of models with a general Lagrangian that includes both parity even and parity odd quadratic terms.
- An important difference between these two classes of PGT models is manifest in their particle spectra.
- Generically, the particle spectrum of the parity conserving PGT model consists of the massless graviton and eighteen massive torsion modes. The conditions for the absence of ghosts and tachyons impose serious restrictions on the propagation of these modes.
- On the contrary a recent analysis of the general PGT shows that the propagation of torsion modes is much less restricted. This is a new and physically interesting dynamical effect of the parity odd sector.

- Based on the experience stemming from general relativity (GR), it is well known that exact solutions play an important role in understanding gravitational dynamics. An important class of these solutions consists of the gravitational waves.
- For many years, investigation of gravitational waves has been an interesting subject also in the framework of PGT, as well as in the metric-affine gravity theory.
- Noticing that dynamical effects of the parity odd sector of PGT are not sufficiently well known, recently YO has studied exact plane wave solutions with torsion in vacuum, propagating on the flat background ($\Lambda = 0$).
- In another recent work of BC and MB complementary results have been obtained, when the generalized *pp* waves with torsion were derived as exact vacuum solutions of the parity even PGT, but for the case of $\Lambda \neq 0$.

- In the present paper, we merge and extend these investigations by constructing the generalized plane waves with torsion as vacuum solutions of the general quadratic PGT with nonvanishing cosmological constant. The resulting structure offers a deeper insight into the dynamical role of the parity odd sector of PGT.
- Our conventions are given as follows. The Latin indices $(i, j, ...)$ refer to the local Lorentz (co)frame and run over $(0, 1, 2, 3)$, b^i is the tetrad (1-form), and h_i is the dual basis (frame), $h_i \rfloor b^k = \delta^i_k$; the volume 4-form is $\hat{\epsilon} = b^0 \wedge b^1 \wedge b^2 \wedge b^3$, the Hodge dual of a form α is ${}^\star\alpha$, with ${}^\star 1 = \hat{\epsilon}$, totally antisymmetric tensor is defined by ${}^\star(b_i \wedge b_j \wedge b_k \wedge b_m) = \varepsilon_{ijkm}$ and normalized to $\varepsilon_{0123} = +1$; the exterior product of forms is implicit.

- Basic dynamical variables of PGT are the tetrad field b^i and the Lorentz (metric compatible) connection ω^{ij} (1-forms), the related field strengths are the torsion $T^i = db^i + \omega^i{}_m b^m$ and the curvature $R^{ij} := d\omega^{ij} + \omega^i{}_m \omega^{mj}$ (2-forms), and the underlying spacetime continuum is characterized by a Riemann-Cartan geometry.
- General PGT dynamics is determined by a locally Poincaré invariant Lagrangian $L_G = L_G(b^i, T^i, R^{ij})$ (4-form). After introducing the covariant gravitational momenta $H_i := \partial L_G/\partial T^i$ and $H_{ij} := \partial L_G/\partial R^{ij}$, the variation of L_G with respect to b^i and ω^{ij} yields the PGT vacum field equations:

$$
\begin{aligned}
&(1ST) \qquad \nabla H_i + E_i = 0\,, \quad E_i := \bar\partial L_G/\bar\partial b^i \\
&(2ND) \qquad \nabla H_{ij} + E_{ij} = 0\,, \quad E_{ij} := \partial L_G/\partial \omega^{ij}\,. \qquad (2.1)
\end{aligned}
$$

- We assume L_G to be at most quadratic function of the field strengths, containing all parity even and parity odd terms,

$$L_G = -{}^\star(a_0 R + \bar{a}_0 X + 2\Lambda) + T^i \sum_{n=1}^{3} \left(a_n {}^{\star(n)}T_i + \bar{a}_n {}^{(n)}T_i \right) + \frac{1}{2} R^{ij} \sum_{n=1}^{6} \left(b_n {}^{\star(n)}R_{ij} + \bar{b}_n {}^{(n)}R_{ij} \right), \qquad (2.2)$$

  where ${}^{(n)}T^i$ and ${}^{(n)}R^{ij}$ are irreducible field strengths, the coupling constants (a_n, b_n) refer to the parity even, and $(\bar{a}_n, \bar{b}_n)$ to the parity odd sector, and Λ is a cosmological constant.
- However, not all of the invariant terms in L_G are independent. One can simplify the calculations by choosing $\bar{a}_2 = \bar{a}_3$ and $\bar{b}_2 = \bar{b}_4$, $\bar{b}_3 = \bar{b}_6$.

- We shall first describe Riemannian *pp* waves in the tetrad formalism, and then introduce a wave ansatz, suitable for transition to the RC geometry of PGT.
- Metric of a subclass of the Kundt class of spacetimes reads

$$
\begin{aligned}
ds^2 &= \left(\frac{q}{p}\right)^2 du(Sdu+dv) - \frac{1}{p^2}(dy^2+dz^2)\,,\\
p &= 1+\frac{\lambda}{4}(y^2+z^2)\,, \quad q = 1-\frac{\lambda}{4}(y^2+z^2)\,,\\
S &= -\frac{\lambda}{4}v^2 + \frac{p}{q}H(u,y,z)\,.
\end{aligned} \tag{3.1}
$$

This metric represents a family of exact solutions of the general relativity with a cosmological constant (GR_Λ), such that for $\lambda \to 0$ it reduces to the *pp* waves in the Minkowski spacetime. We will refer to this family as pp_Λ waves for short.

- Choosing the tetrad field

$$b^0 = du\,,\ b^1 = \frac{1}{2}\left(\frac{q}{p}\right)^2 (Sdu + dv)\ ,\ b^2 = \frac{1}{p}dy\,,\ b^3 = \frac{1}{p}dz\,, \quad (3.2)$$

  the metric can be written in the form $ds^2 = \eta_{ij} b^i \otimes b^j$, where η_{ij} is the half-null Minkowski metric.
- The metric is characterized by the null 1-form $k := du = b^0$, such that $k_i = (1,0,0,0)$ and $k^2 = 0$.
- Using the notation $A = (0,1)$, $a = (2,3)$ the Riemannian connection takes the form

$$\tilde{\omega}^{01} = \frac{\lambda v}{2} b^0 - \frac{\lambda}{q}(yb^2 + zb^3)\,, \qquad \tilde{\omega}^{23} = \frac{\lambda}{2}(zb^2 - yb^3)\,,$$
$$\tilde{\omega}^{Ac} = b^A\left(-\frac{q}{p}\partial^c \frac{p}{q}\right) + k^A \frac{q^2}{2p^2}(\partial^c S)\, b^0\,. \quad (3.3)$$

- Consequently, the only components of curvature that differ from the background values are
$$\tilde{R}^{1c} = -\lambda b^1 b^c + k^1 b^0 Q^c\,, \qquad (3.4)$$
where $Q^c = h_0 \rfloor \tilde{R}^{1c}$ is a 1-form on the wave surface.
- The whole curvature can be represented compactly as
$$\tilde{R}^{ij} = -\lambda b^i b^j + 2b^0 k^{[i} Q^{j]}\,. \qquad (3.5)$$
- The Ricci 1-form $\widetilde{Ric}^i := h_m \rfloor \tilde{R}^{mi}$ reads
$$\begin{aligned} \widetilde{Ric}^i &= -2\lambda b^i + k^i Q b^0\,, \\ Q &= \frac{qp}{2}\left(\partial_{yy} H + \partial_{zz} H + \frac{2\lambda}{p^2} H\right), \end{aligned} \qquad (3.6)$$
and the scalar curvature reads $\tilde{R} = -12\lambda$.
- In GR_Λ, the traceless piece of the field equations takes the well-known form $Q = 0$.

- Our construction of the generalized *pp* waves with torsion is based on the following ansatz.
  (a) The new tetrad field coincides with the Riemannian expression (3.2).
  (b) The new, RC connection is obtained from the Riemannian expression (3.3) by modifying only its radiation piece, according to the rule $\partial^c S \to \partial^c S + K^c$, with $K^A = 0$. Thus,

$$\tilde{\omega}^{Ac} \to \omega^{Ac} = \tilde{\omega}^{Ac} + k^A \frac{q^2}{2p^2} K^c b^0 \,. \qquad (3.7)$$

  Such a construction ensures typical radiation form of the field strengths.
- The geometric meaning of the wave 2-vector K^c is obtained by calculating the torsion,

$$T^i = -k^i \frac{q^2}{2p^2} (b^c K_c) b^0 \,. \qquad (3.8)$$

Generalized *pp* waves with torsion: ansatz

- Going over to the curvature, we find that the RC connection modifies only the radiation piece of the curvature, as expected,

$$R^{1c} = -\lambda b^1 b^c + k^1 b^0 \Omega^c\,, \qquad \Omega^c := Q^c + \Theta^c\,, \tag{3.9}$$

where Θ^c is the term stemming from the torsion.

- The covariant form of the curvature is given by

$$R^{ij} = -\lambda b^i b^j + 2b^0 k^{[i}\Omega^{j]}\,, \tag{3.10}$$

the Ricci curvature reads

$$Ric^i = -3\lambda b^i + b^0 k^i \Omega\,, \qquad \Omega := h_c \rfloor \Omega^c = Q + \Theta\,, \tag{3.11}$$

and the scalar curvature retains its Riemannian form, $R = -12\lambda$.

- The only nonvanishing irreducible part of the torsion is

$$ {}^{(1)}T^i = k^i \frac{q^2}{2p^2} b^0 (b^c K_c)\,. \tag{4.1} $$

- For the curvature, we get ${}^{(3)}R_{ij} = 0$ and ${}^{(5)}R_{ij} = 0$, while

$$ \begin{aligned} &{}^2R^{1c} = \frac{1}{2}{}^\star(\Psi^1 b^c)\,, \quad {}^4R^{1c} = \frac{1}{2}\Phi^1 b^c\,, \quad {}^6R^{ij} = -\lambda b^i b^j\,, \\ &{}^1R^{1c} = R^{1c} - {}^2R^{1c} - {}^4R^{1c} - {}^6R^{1c}\,. \end{aligned} \tag{4.2} $$

- These irreducible components are given in terms of

$$ \begin{aligned} &\Phi^i = k^i(Q+\Theta)b^0\,, \qquad \Psi^i = -k^i\Sigma\, b^0\,, \\ &\Theta := h_c \rfloor \Theta^c = \frac{qp}{2}\left[\partial_y\left(\frac{q}{p}K_y\right) + \partial_z\left(\frac{q}{p}K_z\right)\right], \\ &\Sigma := h_2 \rfloor \Theta_3 - h_3 \rfloor \Theta_2 = \frac{qp}{2}\left[\partial_z\left(\frac{q}{p}K_y\right) - \partial_y\left(\frac{q}{p}K_z\right)\right]. \end{aligned} $$

- To simplify further exposition we introduce the compact notation

$$A_0 = a_0 + \lambda(b_4 + b_6)\,, \qquad A_1 = a_1 + \lambda(b_1 - b_6)\,,$$
$$\bar{A}_0 = \bar{a}_0 - \lambda(\bar{b}_4 - \bar{b}_6)\,, \qquad \bar{A}_1 = \bar{a}_1 + \lambda(\bar{b}_1 - \bar{b}_6)\,.$$

- Using the above nonvanishing irreducible parts of the field strengths, one finds the content of the first PGT field equation

$$\begin{aligned} &3a_0\lambda = \Lambda, \\ &a_1\Theta - A_0(Q + \Theta) + (\bar{a}_1 - \bar{A}_0)\Sigma = 0\,. \end{aligned} \tag{4.3}$$

- There are two nontrivial components of the second field equation, $\mathcal{E}^{1c} := \nabla H^{1c} + E^{1c} = 0$.

$$
\begin{aligned}
&(b_1+b_2)\partial_z(p\Sigma/q)+(b_1+b_4)\partial_y(p\Omega/q)-(a_0-A_1)qK_y/p\\
&+(\bar b_1-\bar b_2)\partial_y(p\Sigma/q)-(\bar b_1-\bar b_4)\partial_z(p\Omega/q)+(\bar a_0-\bar A_1)qK_z/p=0,\\
&-(b_1+b_2)\partial_y(p\Sigma/q)+(b_1+b_4)\partial_z(p\Omega/q)-(a_0-A_1)qK_z/p\\
&+(\bar b_1-\bar b_2)\partial_z(p\Sigma/q)+(\bar b_1-\bar b_4)\partial_y(p\Omega/q)-(\bar a_0-\bar A_1)qK_y/p=0.
\end{aligned}
$$

- One of the possible aproaches is to substitute here the expression for Ω from the first field equation, and solve the resulting system for the field strength functions Θ and Σ.
- The next step would be to deduce the torsion functions K^c and metric function H.
- However, we shall use here an equivalent but technically more direct approach that is based on suitably defined potential variables.

- The RC curvature is defined in terms of the wave surface 1-form $\Omega^c = Q^c + \Theta^c$.
- Introducing the torsion potentials $(V, \bar{V})$ by

$$\frac{q}{p}K_y = \partial_y V + \partial_z \bar{V}\,, \qquad \frac{q}{p}K_z = \partial_z V - \partial_y \bar{V}\,, \tag{4.5}$$

the field strength functions (Ω, Θ, Σ) have the form:

$$\hat{Q} := \frac{p}{q}Q = \frac{1}{2}\left[\hat{\Delta}H + 2\lambda H\right]\,,$$

$$\hat{\Theta} := \frac{p}{q}\Theta = \frac{1}{2}\hat{\Delta}V\,, \qquad \hat{\Sigma} := \frac{p}{q}\Sigma = \frac{1}{2}\hat{\Delta}\bar{V}\,,$$

$$\hat{\Omega} := \frac{p}{q}\Omega = \frac{1}{2}\left[\hat{\Delta}(H + V) + 2\lambda H\right]\,, \tag{4.6}$$

where $\hat{\Delta} = p^2(\partial_{yy} + \partial_{zz})$ is the covariant Laplacian.
- Note that the potentials $(V, \bar{V})$ are not uniquely defined.

- The transformations

$$V \to V + v, \qquad \bar{V} \to \bar{V} + \bar{v}\,,$$

where v and $\bar{v}$ are harmonic functions, $\underline{\Delta} v = \underline{\Delta} \bar{v} = 0$, are unphysical, as they leave the field strengths invariant.

- By integrating the equations stemming from the second field equations and taking into account the unphysical transformations of the torsion potentials we get:

$$(b_1 + b_4)\hat{\Omega} + (\bar{b}_1 - \bar{b}_2)\hat{\Sigma} - (a_0 - A_1)V - (\bar{a}_0 - \bar{A}_1)\bar{V} = 0\,,$$
$$(\bar{b}_1 - \bar{b}_2)\hat{\Omega} - (b_1 + b_2)\hat{\Sigma} - (\bar{a}_0 - \bar{A}_1)V + (a_0 - A_1)\bar{V} = 0\,.$$

- After expressing (Ω, Θ, Σ) in terms of the potentials the PGT field equations become three differential equations for the three unknown functions $(V, \bar{V}, H)$. The solutions of these equations define the complete spacetime geometry associated to the generalized *pp* waves with torsion.

- Torsion field equations can be written in the matrix form as

$$K\hat{\Delta}\mathcal{V} - 2A_0 N\mathcal{V} = 0\,, \qquad \mathcal{V} = (V, \bar{V})^T\,, \tag{5.1a}$$

$$K = \begin{pmatrix} a_1(b_1+b_4) & (\bar{a}_1-\bar{A}_0)(b_1+b_4) + A_0(\bar{b}_1-\bar{b}_2) \\ a_1(\bar{b}_1-\bar{b}_2) & -A_0(b_1+b_2) + (\bar{a}_1-\bar{A}_0)(\bar{b}_1-\bar{b}_2) \end{pmatrix},$$

$$N = \begin{pmatrix} a_0 - A_1 & \bar{a}_0 - \bar{A}_1 \\ \bar{a}_0 - \bar{A}_1 & -(a_0 - A_1) \end{pmatrix}. \tag{5.1b}$$

- For $\det K \neq 0$, one can multiply (5.1a) by K^{-1} and obtain an equation of the Klein-Gordon type:

$$\hat{\Delta}\mathcal{V} - M\mathcal{V} = 0\,, \qquad M = 2A_0 F := 2A_0 K^{-1} N\,, \tag{5.2}$$

where M is interpreted as the mass matrix.

- Equation (5.2) can be solved by diagonalization, that is by finding the eigenvalues and the corresponding eigenvectors of the matrix M.

- The eigenvalues m^2 of M are determined by

$$m^2_{1/2} = \frac{1}{2}\left(\operatorname{tr} M \pm \sqrt{(\operatorname{tr} M)^2 - 4\det M}\right). \qquad (5.3)$$

- If the matrix M has two distinct eigenvalues, one can construct the matrix P that transforms M to its diagonal form $M' := \mathrm{diag}(m_1^2, m_2^2)$.
- Multiplying (5.2) by P^{-1}, one obtains

$$p^2 \underline{\Delta} \mathcal{V}'_n - m_n^2 \mathcal{V}'_n = 0 \qquad (5.4)$$

  where $\mathcal{V}'$ is the eigenvector of M.
- Generic solutions for the potential eigenstates $\mathcal{V}'_n$ are given in terms of the hypergeometric functions ${}_2F_1(a, b, c, z)$.
- Each solution for $\mathcal{V}'_n$ defines the corresponding solution for $\mathcal{V} = (V, \bar{V})^T$. With known $\mathcal{V}_n$, one can determine torsion functions (K_y, K_z).

- In terms of the potentials the first field equation is given by

$$A_0\left(\hat{\Delta}H + 2\lambda H\right) = (a_1 - A_0)\hat{\Delta}V + (\bar{a}_1 - \bar{A}_0)\hat{\Delta}\bar{V}\,. \quad (5.5)$$

  Inserting the solutions for $(V, \bar{V})$ on the rhs this equation becomes an inhomogeneous differential equation for H.
- Its general solution is given by

$$H = H_{\rm h} + H_{\rm p}\,, \quad (5.6)$$

  where $H_{\rm h}$ coincides with the vacuum solution of GR.
- The solution for H obtained by choosing $H_{\rm h} = 0$ has a very interesting interpretation. In that case H reduces just to the particular solution $H_{\rm p}$, the form of which is completely determined by the torsion potentials. Clearly, there are many other solutions for $H_{\rm h}$, and consequently, for H.

- To clarify the role of torsion in our gravitational wave solution, we find it useful to examine the mass spectrum of the associated torsion modes:

$$m^2_{\pm} = \frac{A_0}{\det K}\left(\operatorname{tr} f \pm \sqrt{(\operatorname{tr} f)^2 - 4\det f}\right), \tag{5.7}$$

where f is introduced by $F = f/(\det K)$.

- Since the particle spectrum of PGT has been calculated only with respect to the Minkowski background, a comparison of the result (5.7) to those existing in the literature requires only the values of $m^2_{\pm}$ in the limit $\Lambda \to 0$. Then we have:

$$\begin{aligned} \operatorname{tr} f \; &= \; \Big[-a_1(a_0 - a_1) + (\bar{a}_0 - \bar{a}_1)^2 \Big] (b_1 + b_4) \\ &\quad - a_0(a_0 - a_1)(b_1 + b_2) - 2a_0(\bar{a}_0 - \bar{a}_1)(\bar{b}_1 - \bar{b}_2)\,. \end{aligned}$$

$$\begin{aligned} \det f &= a_0 a_1 \left[(a_0 - a_1)^2 + (\bar{a}_0 - \bar{a}_1)^2 \right] \\ &\times \left[(b_1 + b_2)(b_1 + b_4) + (\bar{b}_1 - \bar{b}_2)^2 \right], \\ \det K &= -a_0 a_1 \left[(b_1 + b_2)(b_1 + b_4) + (\bar{b}_1 - \bar{b}_2)^2 \right]. \end{aligned} \quad (5.8)$$

- To test the obtained formula, we applied it first to the parity even sector of PGT. The corresponding values of $m_{\pm}^2$ are found to coincide with masses of the spin-$2^{\pm}$ torsion modes, known from the literature.
- A more complete verification can be done by comparing with the recent work of G. K. Karananas:
  - The particle spectrum of parity-violating Poincaré gravitational theory, *Class. Quantum Grav.* 32 (2015) 055012.

- Paper by Karananas is the the only existing calculation of the whole mass spectrum for torsion modes in the most general PGT.
- A comparison of the Lagrangian in his paper with our expression is straightforward, although numerous misprints of his complicate the whole process.
- Nevertheless, in the end, we managed to establish the relation between our and Karananas' coupling constants and find that the resulting values of $m^2_\pm$ exactly reproduce the result of Karananas' paper (after correcting his misprints), describing the spin-$2^\pm$ torsion modes.
- Thus, the massive spin-$2^\pm$ torsion modes are an essential ingredient of our gravitational wave.

- We found a new family of the exact vacuum solutions of the most general PGT, with both parity even and parity odd invariants in the Lagrangian, which represents the generalized *pp* waves with torsion.
- Especially important is the ansatz for the Lorentz connection, the radiation piece of which is constructed in terms of the null vector k^i. The torsion and the radiation piece of the curvature satisfy the radiation conditions

$$k^i S_{ij} = 0\,, \qquad \varepsilon^{ijmn} k_j S_{mn} = 0\,, \tag{6.1a}$$

$$k^i T_i = 0\,, \qquad \varepsilon^{ijmn} k_m T_n = 0\,. \tag{6.1b}$$

- The relations (6.1a) represent the well-known Lichnerowicz criterion for identifying gravitational waves, based on analogy with the electromagnetic waves, whereas (6.1b) defines a natural extension to PGT.

- In the limit of vanishing torsion, the generalized *pp* waves with torsion reduce to the pp_Λ waves.
- They are classified as solutions of the Petrov type *N*, since their Weyl tensor satisfies the special algebraic condition $W_{ijmn}k^n = 0$. This criterion can be naturally extended to a RC geometry of PGT as

$$ {}^{(1)}R_{ijmn}k^n = 0\,, \tag{6.2} $$

  where ${}^{(1)}R_{ijmn}$ is the first irreducible part of the curvature tensor. Adopting the criterion (6.2), the generalized *pp* waves with torsion are found to be also of type *N*.
- Of particular importance for the physical interpretation of torsion is the mass matrix *M*, which eigenvalues in the limit $\lambda \to 0$, coincide with the mass squared values of the spin-$2^\pm$ torsion modes.