Integrability of geodesics in contact space $T^{1,1}$ and its metric cone *

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Abstract

We investigate the integrability of geodesics in the five-dimensional Sasaki-Einstein space $T^{1,1}$. We construct explicitly the constants of motion and prove the complete integrability of geodesic motions. This property is also valid for geodesic motions on its Calabi-Yau metric cone. Having in mind that the metric cone is singular at the apex of the cone, we extend the analysis of integrability for resolved conifolds. It is shown that in the case of the *small resolution* the integrability is preserved, while in the case of the *deformation* of the conifold it is lost.

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1. Introduction

Recently there has been much interest in contact geometry in connection with some modern developments in mathematics and theoretical physics [1, 2]. Sasaki-Einstein geometry is considered important in studies of consistent string compactification and in the context of AdS/CFT correspondence [3, 4].

The symmetries of Sasaki-Einstein spaces play a significant role in connection with the study of integrability properties of geodesic motions and separation of variables of the classical Hamilton-Jacobi or quantum Klein-Gordon, Dirac equations.

The homogeneous toric Sasaki-Einstein on $S^2 \times S^3$ is usually referred to as $T^{1,1}$ and was considered as the first example of toric Sasaki-Einstein/quiver duality [5]. The $AdS \times T^{1,1}$ is the first example of a supersymmetric holographic theory based on a compact manifold which is not locally S^5.

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In this paper we are concerned with integrability of geodesic motions in Sasaki-Einstein space $T^{1,1}$ ant its Calabi-Yau metric cone. In general the metric cone of a Sasaki-Einstein manifold is singular at the apex of the cone. On the other hand there are promising generalizations of the original AdS/CFT correspondence by considering $D3$-branes on conifold singularities [5, 6].

In the case of the metric cone over $T^{1,1}$, the singularity at its apex can be smoothed out in two different ways [7]. On can substitute the apex by an S^3 (*deformation*) or by an S^2 (*small resolution*).

In a recent paper [8] it has been constructed the complete set of constants of motion in $T^{1,1}$ space which provides the complete integrability of the geodesic flow. In what follows we extend the investigation of the integrabilty for the geodesic flow on the Calabi-Yau metric cone of $T^{1,1}$ space and its resolved conifolds. We find that the geodesic motions are completely integrable in the case of the metric cone as well as its small resolution of the apex singularity. In opposition to these cases, the geodesic flow on the deformed metric cone is not completely integrable.

The paper is organized as follows. In the next Section we present some mathematical preliminaries regarding the Sasaki-Einstein geometry. In Section 3 we discuss the integrability of geodesic motions in $T^{1,1}$ space and its metric cone. In Section 4 we extend the study of the integrability for the resolved conifolds in both existing procedures of smoothing out the singularity at the apex of the conifold. The paper ends with conclusions in Section 5.

2. Preliminaries regarding the Sasaki-Einstein geometry

Recall that a $(2n-1)$-dimensional manifold M is a contact manifold if there exists a 1-form η, called the contact 1-form, on M such that:

$$\eta \wedge (d\eta)^{n-1} \neq 0\,,$$

everywhere on M [1]. A contact Riemannian manifold with the metric g_M is Sasakian if its metric cone

$$\big(C(M),\, g_{C(M)}\big) = \big(\mathbb{R}_+ \times M\,, dr^2 + r^2 g_M\big)\,, \tag{1}$$

is Kähler with the Kähler form

$$\Omega = \frac{1}{2}\, d(r^2\eta)\,.$$

Here $r \in (0,\infty)$ can be regarded as a coordinate on the positive real line $\mathbb{R}_+$.

As a part of the connection between Sasaki and Kähler geometries, it is worth noting that in the case of a Sasaki-Einstein manifold

$$\mathrm{Ric}_{g_M} = 2(n-1) g_M\,,$$

the metric cone is Ricci flat,

$$\mathrm{Ric}_{g_{C(M)}} = 0\,,$$

i.e. a Calabi-Yau manifold.

3. Complete integrability of $T^{1,1}$ space and its metric cone

The metric of the space $T^{1,1}$ may be written down explicitly by utilizing the fact that it is a $U(1)$ bundle over $S^2 \times S^2$. Let us denote by (θ_1, ϕ_1) and (θ_2, ϕ_2) the coordinates which parametrize the two sphere in a conventional way, and the angle $\psi \in [0, 4\pi)$ parametrizes the $U(1)$ fiber. Using these coordinates the $T^{1,1}$ metric may be written as [7, 9]:

$$\begin{aligned} ds^2_{T^{1,1}} =& \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \\ &+ \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2\,, \end{aligned}$$

with the globally defined contact 1-form η:

$$\eta = \frac{1}{3}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)\,.$$

The Hamiltonian describing the geodesic flow is

$$H = \frac{1}{2} g^{ij}\, p_i p_j\,, \tag{2}$$

where g^{ij} is the inverse metric of $T^{1,1}$ space and $p_i = g_{ij}\,\dot{x}^j$ are the conjugate momenta to the coordinates $(\theta_1, \theta_2, \phi_1, \phi_2, \psi)$. Let us denote by $(p_{\theta_1}, p_{\theta_2}, p_{\phi_1}, p_{\phi_2}, p_\psi)$ these conjugate momenta:

$$\begin{aligned} p_{\theta_1} &= \frac{1}{6}\dot{\theta}_1\,, \\ p_{\theta_2} &= \frac{1}{6}\dot{\theta}_2\,, \\ p_{\phi_1} &= \frac{1}{6}\sin^2\theta_1\,\dot{\phi}_1 + \frac{1}{9}\cos\theta_1(\cos\theta_1\,\dot{\phi}_1 + \cos\theta_2\,\dot{\phi}_2 + \dot{\psi})\,, \\ p_{\phi_2} &= \frac{1}{6}\sin^2\theta_2\,\dot{\phi}_2 + \frac{1}{9}\cos\theta_2(\cos\theta_1\,\dot{\phi}_1 + \cos\theta_2\,\dot{\phi}_2 + \dot{\psi})\,, \\ p_\psi &= \frac{1}{9}(\cos\theta_1\,\dot{\phi}_1 + \cos\theta_2\,\dot{\phi}_2 + \dot{\psi})\,. \end{aligned} \tag{3}$$

In terms of them, Hamiltonian (2) has the form:

$$\begin{aligned} H =& 3\left[p_{\theta_1}^2 + p_{\theta_2}^2 + \frac{1}{\sin^2\theta_1}(p_{\phi_1} - \cos\theta_1\, p_\psi)^2 + \frac{1}{\sin^2\theta_2}(p_{\phi_2} - \cos\theta_2\, p_\psi)^2\right] \\ &+ \frac{9}{2}\, p_\psi^2\,. \end{aligned} \tag{4}$$

Taking into account the isometries of $T^{1,1}$, the momenta $p_{\phi_1}, p_{\phi_2}, p_\psi$ are conserved. On the other hand two total $SU(2)$ angular momenta are also conserved:

$$\begin{aligned}\mathbf{j}_1^2 &= p_{\theta_1}^2 + \frac{1}{\sin^2\theta_1}(p_{\phi_1} - \cos\theta_1\, p_\psi)^2 + p_\psi^2\,,\\ \mathbf{j}_2^2 &= p_{\theta_2}^2 + \frac{1}{\sin^2\theta_2}(p_{\phi_2} - \cos\theta_2\, p_\psi)^2 + p_\psi^2\,.\end{aligned} \tag{5}$$

Other constants of motion can be constructed from the Stäckel-Killing tensors that are admitted by the $T^{1,1}$ space [8]. In spite of the presence of a multitude of conserved quantities, the number of functionally independent constants of motion is five, exactly the dimension of the Sasaki-Einstein space $T^{1,1}$ [8, 10]. This implies the complete integrability of geodesic motions in $T^{1,1}$ which allows us to solve the Hamilton-Jacobi equation by separation of variables and construct the action-angle variables [11].

On the metric cone (1) the geodesic flow is described by the Hamiltonian:

$$H_{C(T^{1,1})} = \frac{1}{2}\, p_r^2 + \frac{1}{r^2}\,\tilde{H}\,, \tag{6}$$

where the radial momentum is

$$p_r = \dot{r}\,.$$

The Hamiltonian $\tilde{H}$ has a similar structure as in (4), but constructed with momenta $\tilde{p}_i$ related to momenta p_i (3) by

$$\tilde{p}_i = r^2 g_{ij}\,\dot{x}^j = r^2\, p_i\,.$$

It is not difficult to observe that the radial dynamics is independent of the dynamics of the base manifold $T^{1,1}$ and $\tilde{H}$ is a constant of motion. The Hamilton equations of motion for $\tilde{H}$ on $T^{1,1}$ have the standard form in terms of a new time variable $\tilde{t}$ given by [12]

$$\frac{dt}{d\tilde{t}} = r^2\,.$$

Concerning the constant of motions, they are the conjugate momenta $(\tilde{p}_{\phi_1}, \tilde{p}_{\phi_2}, \tilde{p}_\psi)$ associated with the cyclic coordinates (ϕ_1, ϕ_2, ψ) and two total $SU(2)$ momenta

$$\tilde{\mathbf{j}}_{1,2}^2 = r^4\, \mathbf{j}_{1,2}^2\,.$$

Together with the Hamiltonian $H_{C(T^{1,1})}$, they ensure the complete integrability of the geodesic flow on the metric cone.

Considering a particular level set $E_{C(T^{1,1})}$ of $H_{C(T^{1,1})}$ we get for the radial motion

$$p_r^2 = \dot{r}^2 = 2E_{C(T^{1,1})} - \frac{2}{r^2}\,\tilde{H}\,.$$

The turning point of the radial motion is determined by

$$\dot{r} = 0 \quad \Longrightarrow \quad r_* = \sqrt{\frac{\tilde{H}}{E_{C(T^{1,1})}}}\,.$$

Projecting the geodesic curves onto the base manifold $T^{1,1}$ we can evaluate the total distance transversed in the Sasaki-Einstein space between the limiting points as $t \to -\infty$ and $t \to +\infty$ [12]

$$d = \sqrt{2\tilde{H}} \int_{-\infty}^{\infty} \frac{dt}{r_*^2 + 2E_{C(T^{1,1})} t^2} = \pi\,.$$

4. Integrability of the resolved conifolds

The conifold (1) associated with Sasaki-Einstein space $T^{1,1}$ can be described by the quadric

$$\sum_{a=1}^{4} w_a^2 = 0\,. \tag{7}$$

on $\mathbb{C}^4$. This equation can be written in terms of a matrix $\mathcal{W}$ defined by

$$\mathcal{W} = \frac{1}{\sqrt{2}}\,\sigma^a w_a = \frac{1}{\sqrt{2}} \begin{pmatrix} w_3 + i w_4 & w_1 - i w_2 \\ w_1 + i w_2 & -w_3 + i w_4 \end{pmatrix} \equiv \begin{pmatrix} X & U \\ V & Y \end{pmatrix},$$

where σ^a are the Pauli matrices for $a = 1, 2, 3$ and σ^4 is i times the unit matrix. The radial coordinate is defined by

$$r^2 = \mathrm{tr}\,(\mathcal{W}^\dagger \mathcal{W})\,.$$

Equation (7) can be written as

$$\det \mathcal{W} = 0 \quad , \quad \text{i.e.} \quad XY - UV = 0\,. \tag{8}$$

The singularity at the apex can be repaired in two different ways. The first is achieved by a *deformation* having the effect of replacing the node by an S^3. The second possibility is represented by a *small resolution* consisting in a replacement of the node by an S^2.

4.1. Small resolution

The small resolution is obtained replacing equation (8) by the pair of equations [7]:

$$\begin{pmatrix} X & U \\ V & Y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0\,,$$

in which $(\lambda_1, \lambda_2) \in \mathbb{CP}^1$ are not both zero.

It turns out to be convenient to introduce a new radial coordinate

$$\rho^2 \equiv \frac{3}{2}\gamma\,,$$

where the function γ is given by the equation

$$\gamma^3 + 6\,a^2\,\gamma^2 - r^4 = 0\,.$$

Here a is the "resolution" parameter representing the radius of the sphere S^2 which replaces the singularity at $r^2 = 0$.

Eventually the metric of the resolved conifold can be written simply as [13]

$$\begin{aligned} ds^2_{rc} = {}& \kappa^{-1}(\rho)\,d\rho^2 + \frac{1}{9}\kappa(\rho)\rho^2(d\psi + \cos\theta_1\,d\phi_1 + \cos\theta_2\,d\phi_2)^2 \\ & + \frac{1}{6}\rho^2(d\theta_1^2 + \sin^2\theta_1\,d\phi_1^2) + \frac{1}{6}(\rho^2 + 6\,a^2)(d\theta_2^2 + \sin^2\theta_2\,d\phi_2^2)\,, \end{aligned} \tag{9}$$

where

$$\kappa(\rho) \equiv \frac{\rho^2 + 9\,a^2}{\rho^2 + 6\,a^2}\,.$$

The resolved conifold metric is Ricci flat and has an explicit $SU(2) \times SU(2)$ invariant form. When the resolution parameter a goes to zero or when $\rho \to \infty$, the resolved conifold metric reduces to the standard conifold metric $g_{C(T^{1,1})}$. In fact, the parameter a introduces an asymmetry between the two sphere.

In order to write the Hamiltonian on the resolved conifold, we evaluate the conjugate momenta $(P_\rho, P_{\theta_1}, P_{\theta_2}, P_{\phi_1}, P_{\phi_2}, P_\psi)$ corresponding to the coordinates $(\rho, \theta_1, \theta_2, \phi_1, \phi_2, \psi)$:

$$\begin{aligned} P_\rho &= \kappa^{-1}(\rho)\dot\rho\,, \\ P_{\theta_1} &= \frac{1}{6}\rho^2\dot\theta_1\,, \\ P_{\theta_2} &= \frac{1}{6}(\rho^2 + 6\,a^2)\dot\theta_2\,, \\ P_{\phi_1} &= \frac{1}{6}\rho^2\sin^2\theta_1\,\dot\phi_1 + \frac{1}{9}\kappa(\rho)\rho^2\cos\theta_1(\cos\theta_1\,\dot\phi_1 + \cos\theta_2\,\dot\phi_2 + \dot\psi)\,, \\ P_{\phi_2} &= \frac{1}{6}(\rho^2 + 6\,a^2)\sin^2\theta_2\,\dot\phi_2 + \frac{1}{9}\kappa(\rho)\rho^2\cos\theta_2(\cos\theta_1\,\dot\phi_1 + \cos\theta_2\,\dot\phi_2 + \dot\psi)\,, \\ P_\psi &= \frac{1}{9}\kappa(\rho)\rho^2(\cos\theta_1\,\dot\phi_1 + \cos\theta_2\,\dot\phi_2 + \dot\psi)\,. \end{aligned}$$

In terms of them, the Hamiltonian for the geodesic flow on resolved conifold (rc) is

$$\begin{aligned} H_{rc} = & \frac{\kappa(\rho)}{2} P_\rho^2 + \frac{9}{2} \frac{1}{\kappa(\rho)\,\rho^2} P_\psi^2 + \frac{3}{\rho^2} P_{\theta_1}^2 + \frac{3}{\rho^2 + 6\,a^2} P_{\theta_2}^2 \\ & + \frac{3}{\rho^2 \sin^2\theta_1} (P_{\phi_1} - \cos\theta_1\, P_\psi)^2 \\ & + \frac{3}{(\rho^2 + 6\,a^2)\sin^2\theta_2} (P_{\phi_2} - \cos\theta_2\, P_\psi)^2 \,. \end{aligned} \tag{10}$$

We observe that (ϕ_1, ϕ_2, ψ) are still cyclic coordinates and accordingly, momenta $P_{\phi_1}, P_{\phi_2}, P_\psi$ are conserved. Taking into account the $SU(2) \times SU(2)$ symmetry of the metric (9), the total angular momenta

$$\begin{aligned} \mathbf{J}_1^2 &= P_{\theta_1}^2 + \frac{1}{\sin^2\theta_1}(P_{\phi_1} - \cos\theta_1\, P_\psi)^2 + P_\psi^2 = \rho^4\,\mathbf{j}_1^2\,, \\ \mathbf{J}_2^2 &= P_{\theta_2}^2 + \frac{1}{\sin^2\theta_2}(P_{\phi_2} - \cos\theta_2\, P_\psi)^2 + P_\psi^2 = (\rho^2 + 6a^2)^2\,\mathbf{j}_2^2\,. \end{aligned}$$

are also conserved. Using these total angular momenta, the Hamiltonian H_{rc} can be put in the form:

$$H_{rc} = \frac{1}{2}\frac{\rho^2 + 9\,a^2}{\rho^2 + 6\,a^2} P_\rho^2 + \frac{3}{\rho^2}\mathbf{J}_1^2 + \frac{3}{\rho^2 + 6\,a^2}\mathbf{J}_2^2 - \frac{3(\rho^2 + 12\,a^2)}{2(\rho^2 + 6\,a^2)(\rho^2 + 9\,a^2)} P_\psi^2 \,. \tag{11}$$

The set of conserved quantities $(H_{rc}, P_{\phi_1}, P_{\phi_2}, P_\psi, \mathbf{J}_1^2, \mathbf{J}_2^2)$ ensure the complete integrability of geodesic motions on the resolved conifold. As it is expected, for $a = 0$ we recover the state of integrability on the standard metric cone of the Sasaki-Einstein space $T^{1,1}$.

Considering a particular level set E_{rc} of H_{rc}, we can integrate (11) for ρ and the turning point ρ_* is determined by

$$\dot\rho = \kappa(\rho) P_\rho = 0\,.$$

The explicit expression of the turning point ρ_* is quite involved and is not produced here.

4.2. Deformation

The deformation of the conifold consists in replacing the apex by an S^3 which is achieved by another modification of equation (7). The deformed conifold is describe by the equation:

$$\sum_{a=1}^{4} w_a^2 = \epsilon^2\,,$$

where ϵ is the "deformation" parameter.

On setting the new radial coordinate

$$r^2 = \epsilon^2 \cosh\tau\,,$$

the deformed conifold (dc) metric is [7, 14]:

$$ds^2_{dc} = K\epsilon^{\frac{4}{3}}\left(\frac{\sinh^3\tau}{3\,(\sinh 2\tau - 2\tau)}\,(d\tau^2 + ds_1^2) + \frac{\cosh\tau}{4}\,ds_2^2 + \frac{1}{4}\,ds_3^2\right)\,, \quad (12)$$

where

$$K(\tau) = \frac{(\sinh 2\tau - 2\tau)^{\frac{1}{3}}}{2^{\frac{1}{3}}\,\sinh\tau}\,,$$

and

$$\begin{aligned} ds_1^2 &= (d\psi + \cos\theta_1\,d\phi_1 + \cos\theta_2\,d\phi_2)^2\,,\\ ds_2^2 &= d\theta_1^2 + d\theta_2^2 + \sin^2\theta_1\,d\phi_1^2 + \sin^2\theta_2\,d\phi_2^2\,,\\ ds_3^2 &= 2\big(\sin\psi(d\phi_1\,d\theta_2\,\sin\theta_1 + d\phi_2\,d\theta_1\,\sin\theta_2)\\ &\qquad + \cos\psi(d\theta_1\,d\theta_2 - d\phi_1\,d\phi_2\,\sin\theta_1\,\sin\theta_2)\big)\,. \end{aligned}$$

We remark that the metric (12) of the deformed conifold is more involved than in the precedent case. We have only two cyclic coordinates ϕ_1 and ϕ_2 and the number of the first integrals of the corresponding Hamiltonian is insufficient to ensure the integrability of the geodesic motions.

5. Conclusions

In the last time it was proved the complete integrability of geodesic motions on five-dimensional Sasaki-Einstein spaces $Y^{p,q}$ and $T^{1,1}$ [8, 10, 15].

The purpose of this paper was to investigate the integrability in the case of the metric cone of $T^{1,1}$ space and its resolved conifolds. We proved that the geodesics on the Calabi-Yau metric cone are also completely integrable. This property is also valid for the small resolution of the conifold, but it is lost in the case of the deformation.

It would be interesting to look for an action-angle formulation of these completely integrable systems [11, 16]. We mention that in the case of the geodesic flow on metric cones there are some subtle points having in mind that the radial motion is unbounded. Moreover it is of interest to study the integrability in higher dimensional Sasaki-Einstein spaces and their non-singular resolutions, relevant for the predictions of Ads/CFT correspondence.

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