Intrinsic non-commutativity in quantum gravity*

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Abstract

We summarize our recent work on the foundational aspects of string theory as a quantum theory of gravity. We emphasize the hidden quantum geometry (modular spacetime) behind the generic representation of quantum theory and then stress that the same geometric structure underlies a manifestly T-duality covariant formulation of string theory, that we call metastring theory. We also discuss the effective non-commutative description of closed strings as implied by an intrinsic non-commutativity of closed string theory. This fundamental non-commutativity is explicit in the metastring formulation of quantum gravity.

1. Introduction

In this talk we outline the essence of our recent work [1–8] on the foundational aspects of quantum gravity in the form of string theory. In the first section we describe the hidden quantum spacetime geometry underlying the generic representation of quantum theory (which renders it manifestly non-local) and then in second section we find that the same (and, in general, dynamical) geometric structure underlies metastring theory, a manifestly T-duality covariant formulation of string theory. Thus quantum gravity "gravitizes" the quantum spacetime geometry. Finally, in the last section we outline the effective description of closed strings at long distance that leads to a non-commutative effective field theory, as implied by an intrinsic non-commutativity of closed string theory, and we discuss some of its consequences.

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2. Quantum theory and quantum spacetime

We start by revealing the hidden quantum spacetime geometry of quantization [5], which will, surprisingly, take us all the way to quantum gravity. We focus on the Heisenberg (or Weyl-Heisenberg) group, which is generated, on the level of the corresponding algebra, by familiar position $\hat{q}^a$ and momentum $\hat{p}_b$ operators:

$$[\hat{q}^a, \hat{p}_b] = i\hbar\delta^a_b. \tag{1}$$

It will be convenient to introduce a length scale λ and a momentum scale ϵ, with $\lambda\epsilon = \hbar$. Then, let us introduce the following notation $\hat{x}^a \equiv \hat{q}^a/\lambda, \quad \hat{\tilde{x}}_a \equiv \hat{p}_a/\epsilon,$, with $[\hat{x}^a, \hat{\tilde{x}}_b] = i\delta^a_b$. Even more compactly let us suggestively write

$$\mathbb{X}^A \equiv (x^a, \tilde{x}_a)^T, \quad [\hat{\mathbb{X}}^a, \hat{\mathbb{X}}^b] = i\omega^{AB}, \tag{2}$$

with $\frac{1}{2}\omega_{AB}dX^A dX^B = \frac{1}{\hbar}dp_a \wedge dq^a$, where $\omega_{AB} = -\omega_{BA}$ is the canonical symplectic form on phase space $\mathcal{P}$. The Heisenberg group $H_\mathcal{P}$ is generated by Weyl operators [9] ($\mathbb{K}$ stands for the pair $(\tilde{k}, k)$ and $\omega(\mathbb{K}, \mathbb{K}') = k \cdot \tilde{k}' - \tilde{k} \cdot k'$)

$$W_\mathbb{K} \equiv e^{2\pi i\omega(\mathbb{K},\mathbb{X})}. \tag{3}$$

These form a central extension of the translation algebra

$$W_\mathbb{K} W_{\mathbb{K}'} = e^{2\pi i\omega(\mathbb{K},\mathbb{K}')} W_{\mathbb{K}+\mathbb{K}'}. \tag{4}$$

The projection $\pi : H_\mathcal{P} \to \mathcal{P}$ (where $\pi : W_\mathbb{K} \to \mathbb{K}$) defines a line bundle over $\mathcal{P}$ (in principle a covariant phase space of quantum probes). In this formulation, states are sections of degree one

$$W_{\mathbb{K}'}\Phi(\mathbb{K}) = e^{2\pi i\omega(\mathbb{K},\mathbb{K}')}\Phi(\mathbb{K} + \mathbb{K}'). \tag{5}$$

In this language, geometric quantization means to take a Lagrangian $L \in \mathcal{P}$, so that states descend to square integrable functions on L.

A Lagrangian submanifold L is a maximally isotropic subspace L with $\omega|_L = 0$, and thus $\{\partial/\partial q^a\} \in T\mathcal{P}$ defines a Lagrangian submanifold, or "space". (Indeed, $\omega(\partial/\partial q^a, \partial/\partial q^b) = 0$.) This can be understood as a classical characterization of space (and in the covariant context, of spacetime), as a "slice" of phase space. How about a purely quantum characterization of space? We claim that quantum theory reveals a new notion of quantum space (and, more covariantly, a new notion of quantum spacetime).

Note that for space-like separations the operators of a local quantum field theory commute. Thus in order to understand the meaning of quantum spacetime (quantum Lagrangian), we need to look at a maximally commuting subalgebra of the Heisenberg algebra and the representation that diagonalizes it. Thus, borrowing from notions of non-commutative algebra and non-commutative geometry [10] (such as the theorem of Gelfand-Naimark [11]), we can say that a Lagrangian submanifold is a maximally commutative subgroup of the Heisenberg group. If we accept this notion of a Lagrangian, then the quantum regime is very different from the classical regime. In particular the vanishing Poisson bracket $\{f(q), g(p)\}$ requires either f or g to be constant. However, the vanishing commutator $[f(\hat{q}), g(\hat{p})] = 0$ requires only that the functions be commensurately periodic

$$e^{i\alpha\hat{p}}e^{i\beta\hat{q}} = e^{i\hbar\alpha\beta}e^{i\beta\hat{q}}e^{i\alpha\hat{p}}, \quad \alpha\beta = 2\pi/\hbar. \tag{6}$$

What is interesting here is that similar considerations led Aharonov to introduce *modular variables* to describe purely quantum phenomena, such as interference [12] (see also the prescient work of Schwinger [13]).

2.1. Modular variables

Modular variables are described in great detail in the very insightful book by Aharonov and Rohrlich [12], where one can find detailed bibliography on this subject[1]. The fundamental question posed there was as follows: how does one capture interference effects (due to the fundamental linearity of quantum theory) in terms of Heisenberg operators? For example, what are the quantum observables that can measure the relative phase responsible for interference in a double-slit experiment? No polynomial functions of the operators $\hat{q}$ and $\hat{p}$ can detect such phases, but operators that translate in space, such as $e^{iR\hat{p}/\hbar}$, do. Thus the modular variables denoted $[\hat{q}]$ and $[\hat{p}]$, which are defined modulo a length scale R (the slit spacing being a natural choice), play a central role, where

$$[p]_{h/R} = p \mod (h/R)\,, \qquad [q]_R = q \mod (2\pi R)\,, \tag{7}$$

and $h = 2\pi\hbar$. The shift operator $e^{iR\hat{p}/\hbar} = e^{iR[\hat{p}]/\hbar}$ shifts the position of a particle state (say an electron in the double-slit experiment) by a distance R and is a function of the modular momenta. These modular variables (the main examples being the Aharonov-Bohm and Aharonov-Casher phases [12]) satisfy non-local operator equations of motion. For example, given the Hamiltonian, $\hat{H} = \hat{p}^2/2m + V(\hat{q})$, the Heisenberg equation of motion for the shift operator is,

$$e^{-iR\hat{p}/\hbar}\frac{d}{dt}e^{iR\hat{p}/\hbar} = -\frac{iR}{\hbar}\left(\frac{V(\hat{q}+R) - V(\hat{q})}{R}\right). \tag{8}$$

Modular variables are fundamentally non-local in a non-classical sense, since we see here that their evolution depends on the value of the potential at distinct locations. Remarkably, thanks to the uncertainty principle, this dynamical non-locality does not lead to a violation of causality [12]. One of the characteristic features of these variables is that they do not have

[1] See also [13–15].

classical analogues; indeed, the limit $\hbar \to 0$ of $[p]_{h/R}$ is ill-defined. Also modular variables capture entanglement of continuous q, p variables.

Note that modular variables are, in general, covariant and, also, contextual[2]. In other words, they carry specific experimental information, such as the length R between the two-slits. However, in the context of quantum gravity such scales are automatically built in, and the contextuality is in principle removed. Also, the fundamental dynamical equations for modular variables are non-local in quantum gravity because of the presence of the fundamental length.

When exponentiated (i.e. when understood as particular Weyl operators), the modular variables naturally commute. In other words, given $[x^a, \tilde{x}_b] = \frac{i}{2\pi}\delta^a_b$, the following commutator of modular operators vanishes [5]

$$[e^{2\pi i x}, e^{2\pi i \tilde{x}}] = 0. \tag{9}$$

Thus a quantum algebra of modular variables possesses more commutative directions than a classical Poisson algebra, because the Poisson bracket of modular variables does not vanish, $\{e^{2\pi i x}, e^{2\pi i \tilde{x}}\} \neq 0$.

Here we make a historical note [16]: The above non-local equations of motion were essentially written by Max Born, in the very first paper which used the phrase "Quantum Mechanics" in its title, in 1924, one year before the Heisenberg breakthrough paper. Actually, Heisenberg crucially used Born's prescription of replacing classical equations by the corresponding difference equations, in order to derive what we now call the canonical commutation relations (properly written by Born and Jordan) from the Bohr-Sommerfeld quantization conditions.

2.2. Modular space and geometry of quantum theory

Returning to the subject of quantum Lagrangians, note that the quantum Lagrangian is analogous to a Brillouin cell in condensed matter physics. The volume and shape of the cell are given by λ and ϵ (i.e. $\hbar$ and G_N (α')) The uncertainty principle is implemented in a subtle way: we can specify a point in modular cell, but if so, we can't say *which* cell we are in.

This means that there is a more general notion of quantization [5], beyond that of geometric quantization. Instead of selecting a classical polarization L (the arguments of the wave function, or the arguments of a local quantum field) we can choose a *modular polarization.* In terms of the Heisenberg group all that is happening is that in order to have a commutative algebra, we need only

$$\omega(\mathbb{K}, \mathbb{K}') \in 2Z, \quad W_{\mathbb{K}} W_{\mathbb{K}'} = e^{2\pi i \omega(\mathbb{K}, \mathbb{K}')} W_{\mathbb{K}+\mathbb{K}'} = W_{\mathbb{K}'} W_{\mathbb{K}'}. \tag{10}$$

[2] Aharonov and collaborators have pushed the logic associated with modular variables to argue for a new kind of weak measurements of such non-local variables that capture the superposition principle of quantum theory. Similarly, Aharonov and collaborators argue for a time symmetric formulation of quantum theory [12].

This defines a lattice Λ in phase space $\mathcal{P}$. Finally, we specify a "lift" of the lattice from the phase space $\mathcal{P}$ to the Heisenberg group $H_{\mathcal{P}}$.

Maximally commuting subgroups $\hat{\Lambda}$ of the Heisenberg group correspond to lattices that are integral and self-dual with respect to ω [17, 18]. Given W_λ where $\lambda \in \Lambda$ there is a lift to $\hat{\Lambda}$ which defines "modular polarization"

$$U_\lambda = \alpha(\lambda) W_\lambda, \tag{11}$$

where $\alpha(\lambda)$ satisfies the co-cycle condition

$$\alpha(\lambda)\alpha(\mu)e^{\pi i \omega(\lambda,\mu)} = \alpha(\lambda+\mu), \quad \lambda, \alpha \in \Lambda. \tag{12}$$

One can parametrize a solution to the co-cycle condition by introducing a symmetric bilinear from η and setting (with $\eta(\mathbb{K}, \mathbb{K}') = k \cdot \tilde{k}' + \tilde{k} \cdot k'$,)

$$\alpha_\eta(\lambda) \equiv e^{i\frac{\pi}{2}\eta(\lambda,\lambda)}. \tag{13}$$

Finally, when we choose a classical Lagrangian L, there is a special state that we associate with the vacuum: it is translation invariant (which in our context can be interpreted as "empty space"). In modular quantization there is no such translation invariant state (because of the lattice structure). The best we can do is to choose a state that minimizes an "energy", which requires the introduction of another symmetric bilinear form, that we call, again suggestively, H. This means, first, that we are looking for operators such that

$$[\hat{\mathbb{P}}_A, \Phi] = \frac{i}{2\pi}\partial_A \Phi, \qquad \Phi(\hat{\mathbb{X}} + \lambda) = \Phi(\hat{\mathbb{X}}), \tag{14}$$

where the modular observables $\Phi(\hat{\mathbb{X}} + \lambda) = \Phi(\mathbb{X})$ are generated by the lattice observables U_λ with $\lambda \in \Lambda$. Translation invariance would be the condition $\hat{\mathbb{P}}|0\rangle = 0$. Since this is not possible, the next natural choice is to minimize the translational energy. Therefore we pick a positive definite metric H_{AB} on $\mathcal{P}$, and we define [5]

$$\hat{E}_H \equiv H^{AB}\hat{\mathbb{P}}_A\hat{\mathbb{P}}_B, \tag{15}$$

and demand that $|0\rangle_H$ be the ground state of $\hat{E}_H$. This is indeed the most natural choice and it shows that we cannot fully disentangle the kinematics (i.e., the definition of translation generators) from the dynamics. In the Schrödinger case, since the translation generators commute, the vacuum state $\hat{E}|0\rangle = 0$ is also the translation invariant state and it carries no memory of the metric H needed to define the energy. In our context, due to the non-commutativity of translations, the operators $\hat{E}_H$ and $\hat{E}_{H'}$ do not commute. As a result the vacuum state depends on H, in other words $|0\rangle_H \neq |0\rangle_{H'}$, and it also possesses a non-vanishing zero point energy.

Thus, modular quantization involves the introduction of three quadratic forms (ω, η, H), i.e. what we call *Born geometry* [1,2], which underlies the geometry of modular variables. As we will see, in the context of metastring theory, a choice of polarization is a choice of a spacetime within $\mathcal{P}$ but the most general choice is a *modular polarization* that we have discussed above. From the foundational quantum viewpoint Born geometry (ω, η, H) arises as a parametrization of such quantizations, which results in a notion of quantum spacetime, that we call *modular spacetime*. Finally, large spacetimes of canonical string theory and general relativity result as a "many-body" phenomenon, through a process of tensoring of unit modular cells, that we refer to as "extensification" [5].

In particular, the symplectic structure ω $ds^2_\omega = \frac{1}{2}\omega_{AB} d\mathbb{X}^A d\mathbb{X}^B = \frac{1}{\hbar} dp_a \wedge dq^a$, is encoded in the canonical Heisenberg commutator between q^a and p_a. The generalized, quantum, metric H comes from the Born rule in quantum theory $ds^2_H = H_{AB} d\mathbb{X}^A d\mathbb{X}^B = \frac{1}{\hbar}(\frac{dq_a dq^a}{G_N} + G_N dp_a dp^a)$. For weak gravity, this metric reduces to the spacetime metric (where spacetime can be viewed as a slice of phase space). Due to gravity's extreme weakness, we only see spacetime metric at low energies. (The ratio ϵ/λ defines a tension; if this is identified with c^3/G_N, it is enormous, $\sim 10^{32} kg/sec$.) Therefore, in this formulation the usual dynamical spacetime metric is the low energy leftover of the quantum metric. Finally, the polarization (or locality metric) η encodes the distinction between spacetime-like and energy-momentum-like aspects of phase space (and in this sense it defines an analog of the "causal" structure in phase space) $ds^2_\eta = \eta_{AB} d\mathbb{X}^A d\mathbb{X}^B = \frac{2}{\hbar} dp_a dq^a$. This new metric captures the essence of relative locality - when η is constant we have absolute locality. Curving η also means "gravitizing the quantum". In general all three elements of Born geometry, ω, η and H are dynamical and curved in metastring theory, as we will discuss in what follows.

Also, we have that the Lorentz group (in D spacetime dimensions) lies at the intersection of the symplectic, neutral and doubly orthogonal groups [5],

$$\mathrm{O}(1, d-1) = \mathrm{Sp}(2d) \cap \mathrm{O}(d, d) \cap \mathrm{O}(2, 2(d-1)). \tag{16}$$

which sheds new light on the origin of quantum theory through compatibility of the causal (Lorentz) structure and the non-locality captured by the discreteness of the quantum spacetime. This also captures the role of relative (observer-dependent) locality [19] needed to resolve the apparent contradiction between discreteness of quantum spacetime and Lorentz symmetry.

Before doing so, let us end by a few comments regarding the Stone-von Neumann theorem [20–22] which asserts that all representations of the Heisenberg group are unitarily equivalent. Normally, we think of this as a choice of basis in phase space (a choice of polarization or classical Lagrangian), and all such choices are related by Fourier transform. Similarly, one can pass from a classical polarization (such as the Schrödinger representation) to a modular polarization via the Zak transform [23]. Note that, there is a connection on the line bundle over phase space that has unit flux through a modular cell. (This is very similar to integer quantum Hall effect systems.) A modular wave function is quasi-periodic

$$\Psi(x+a,\tilde{x}) = e^{2i\pi a\tilde{x}}\Psi(x,\tilde{x}), \quad \Psi(x,\tilde{x}+\tilde{a}) = \Psi(x,\tilde{x}). \tag{17}$$

The quasi-periods correspond to the tails of an Aharonov-Bohm [24] potential attached to a unit flux. In particular, vacuum states must have at least one zero in a cell, which leads to theta functions (the Zak transforms of Gaussians). Note that from the point of modular polarization, the familiar Schrödinger polarization is just a singular limit.

3. From modular spacetime to quantum gravity

The unexpected outcome of our research is that this fundamental quantum geometry of quantum theory can be realized in the context of metastring theory, where this quantum geometry is "gravitized" (i.e. dynamical). At the classical level, metastring theory [1–8] can be thought of as a formulation of string theory in which the target space is doubled in such a way that T-duality acts linearly on the coordinates. This doubling means that momentum and winding modes appear on an equal footing. We refer to the target space as a phase space since the metastring action requires the presence of a background symplectic form ω. The metastring formulation also requires the presence of geometrical structures that generalize to phase space the spacetime metric and the B-field (where the B-field originates from the symplectic structure ω). In fact, in the metastring we have not one but *two* notions of a metric. The first metric η is a neutral metric that defines a bi-Lagrangian structure and allows to define the classical spacetime as a Lagrangian sub-manifold[3] — more precisely, the classical spacetime is defined as a null subspace for η which is also Lagrangian for ω. The second metric H is a metric of signature $(2, 2(D-1))$ that encodes the geometry along the classical spacetime (of dimension D) as well as the transverse energy-momentum space geometry. In this formulation, T-duality exchanges the Lagrangian sub-manifold with its image under $J = \eta^{-1}H$. Classical metastring theory is defined by the following action [3] (see also the pioneering papers [25–27])

$$\hat{S} = \frac{1}{4\pi}\int_{\Sigma} d^2\sigma\Big(\partial_\tau \mathbb{X}^A(\eta_{AB}+\omega_{AB})(\mathbb{X})\partial_\sigma\mathbb{X}^B - \partial_\sigma\mathbb{X}^A H_{AB}(\mathbb{X})\partial_\sigma\mathbb{X}^B\Big), \tag{18}$$

[3] We remind the reader that in symplectic geometry, a Lagrangian subspace is a half-dimensional submanifold of phase space upon which the symplectic form pulls back to zero. In simple terms, a Lagrangian submanifold might be the subspace coordinatized by the q's within the phase space coordinatized by q's and p's.

where $\mathbb{X}^A$ are dimensionless coordinates on phase space and the fields η, H, ω are all dynamical (i.e., in general dependent on $\mathbb{X}$) phase space fields. In the context of a flat metastring we have constant η_{AB}, H_{AB} and ω_{AB}

$$\eta_{AB} \equiv \begin{pmatrix} 0 & \delta \\ \delta^T & 0 \end{pmatrix}, \quad H_{AB} \equiv \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}, \quad \omega_{AB} = \begin{pmatrix} 0 & \delta \\ -\delta^T & 0 \end{pmatrix}, \quad (19)$$

where δ^μ_ν is the d-dimensional identity matrix and $h_{\mu\nu}$ is the d-dimensional Lorentzian metric, T denoting transpose.

In this formulation [3] it is convenient, as suggested by the double field formalism [28,29], to introduce dimensionless coordinates $\mathbb{X}^A \equiv (X^\mu/\lambda, P_\mu/\varepsilon)^T$ on phase space, or equivalently, $\mathbb{X}^A \equiv (x^a, \tilde{x}_a)^T$, where λ and ϵ represent the fundamental spacetime and energy-momentum scales. As already stated, $\hbar = \lambda\epsilon$ and $\alpha' = \frac{\lambda}{\epsilon}$. Given a pair (H, η) it is natural to consider the operator $J \equiv \eta^{-1}H$. The consistency of string theory requires J to be a chiral structure, that is, a real structure ($J^2 = 1$) compatible with η, implying that J is an $O(D, D)$ transformation (realizing generalized T-duality in target space). These three structures, the symplectic $Sp(2D)$ ω, the $O(D, D)$ η and the $SO(2, 2(D-1))$ H, define the new concept of Born geometry [1–5] (see also [30]) which unifies the complex geometry of quantum theory with the metrical geometry of general relativity and the symplectic geometry of canonical Hamiltonian dynamics [31–35]. Note that in the phase space formulation the local phase space coordinates $\mathbb{X}$ are *quasiperiodic* $\mathbb{X}^A(\sigma + 2\pi) = \mathbb{X}^A(\sigma) + \Delta^A$, where Δ^A is the corresponding quasiperiod (which either vanishes for the canonical Polyakov string or is given by the winding number in the usual treatment of T-duality on compact spaces.)

The worldsheet formulation of the metastring is chiral. Thus, even though the fields are doubled the central charges (left and right) are $c_L = c_R = D$ and we still have $D = 26$ for criticality. The metastring is not manifestly invariant under the worldsheet Lorentz transformations and it contains monodromies $\mathbb{X}^A(\sigma + 2\pi) = \mathbb{X}^A(\sigma) + \Delta^A$. The usual Polyakov string can be obtained by integrating out the dual $\tilde{X}$, for constant η and H backgrounds, and by supposing that the monodromies are in the kernel of $(\eta - \omega)$. T-duality is implemented in target space by the action of the chiral J operator ($J \equiv \eta^{-1}H$, $J^2 = 1$): $\mathbb{X} \to J(\mathbb{X})$.

The target space of the metastring is not spacetime, but, to first order, a chiral phase space $\mathcal{P}$ equipped by the symplectic structure ω, and the bilagrangian structure, and in particular, the polarization metric η which relates to the symplectic connection of the Fedosov deformation quantization [36] and thus leads to the star product of deformation quantization, and finally, the quantum H metric which relates to the complex structure in the context of geometric quantization [37], leading to the concept of Hilbert spaces. This classical Born geometry implements the ideas of Born duality in string theory [1, 2].

The classical equations of motion of the metastring $\partial_\tau \mathbb{X}^A - (J\partial_\sigma \mathbb{X})^A = 0$, implies the relation between momenta and monodromies $2\pi P = J(\Delta)$. There is *soldering* between worldsheet null coordinates $\sigma^\pm \equiv \sigma \pm \tau$ and the chiral target space structure $\partial_\pm \mathbb{X}^A - (P_\pm \mathbb{X})^A = 0$, where the chiral projector is defined as $2P_\pm = (1 \pm J)$. This allows us to liberate the left geometry from the right geometry (which is reminiscent of twistor theory). The careful analysis of the metastring action [3] shows that its symplectic form is $\Omega = \frac{1}{4\pi}\int \delta\mathbb{X}^A \eta_{AB} \nabla_\sigma \delta\mathbb{X}^B$, where ∇ is the generalized Fedosov connection found in the Fedosov deformation quantization approach [36].

Also, the operator product expansion of the metastring vertex operators $V_k = \epsilon_k e^{i\mathbb{K}\mathbb{X}}$, (i.e. modular variables) lead to the restriction of $\mathbb{K}$ on a double Lorentzian integral lattice Γ, that by modular invariance, must be self-dual. These exist in $D = 2mod(8)$, and are unique. Criticality gives a very unique lattice $\Gamma = \Pi_{1,25} \times \Pi_{1,25}$. This fact, in turn, leads to the large symmetry structure found by Borcherds in the study of the monstrous moonshine [38, 39][4].

As already noted, the metastring is chiral. This requires the introduction of a preferred worldsheet time coordinate which is fundamentally Lorentzian [3]. How can this be consistent with modular invariance? The answer is given by employing the Giddings-Wolpert-Krichever-Novikov construction [41, 42]: given a Riemann surface, provided a choice of local coordinates around punctures is labeled by one scalar, there exists a unique Abelian differentional e with imaginary periods. The real part of this Abelian differential is the modular invariant time $\tau = Re(e)$. The zeros of e represent interaction points where the worldsheet Lorentzian cones double. Cutting the Riemann surface along the real trajectory of e we obtain a string decomposition of the surface. The Nakamura graphs [43] encode this decomposition and give a very effective cell decomposition of moduli space. Thus Nakamura graphs are the natural Feynman diagrams for closed strings [44].

Finally, the metastring formulation points to an unexpected fundamental non-commutativity of closed string theory, that we address in what follows.

3.1. Intrinsic non-commutativity in metastring theory

It is well established that the structure of the zero mode algebra of the compactified closed string depends on a lattice of momenta $(\Lambda, 2\eta)$ which is integral and self-dual with respect to a neutral metric: a so-called Narain lattice [45]. In our recent work [1–8] we have refined this structure and we have shown that in fact the kinematical structure of the string zero modes depends on a *para-hermitian lattice*: a triple (Λ, η, ω), where Λ is a subgroup of $\mathbb{R}^{2d}$ that describes the lattice of wave-covectors $\lambda\mathbb{K}$, with λ the string length, η is a neutral metric, a symmetric bilinear form of signature

[4]For a string theory related discussion, see [40].

(d,d), and ω is an invertible two-form. This structure needs to satisfy two compatibility conditions: first, the lattice Λ is assumed to be integral with respect to the para-hermitian structure, i.e., $(\eta \pm \omega)(\lambda\mathbb{K}, \lambda\mathbb{K}') \in \mathbb{Z}$, for $\lambda\mathbb{K}, \lambda\mathbb{K}' \in \Lambda$. Second, the metric η and the 2-form ω must be compatible, in the sense that $\eta^{-1}\omega := K$ is a product structure, that satisfies the condition $K^2 = 1$.

These two conditions are a consequence of mutual locality on the worldsheet (i.e. worldsheet causality). It is clear that if (Λ, η, ω) is a para-hermitian lattice, then $(\Lambda, 2\eta)$ is a Narain lattice, so the kinematical structure that we highlight is a refinement of the usual one. The extra information is contained in the 2-form ω. This form does not enter expressions for the spectrum or the partition function and this why it is usually ignored. It does enter however crucially in the definition of the vertex operator algebra and parameterizes what is usually referred to as a cocycle. The role of ω is to promote the zero mode double space $\mathcal{P} \simeq \mathbb{R}^{2d}$ dual to $\mathbb{R}[\Lambda]$ to the status of phase space: $\mathcal{P}$ should be viewed as a symplectic manifold. At the quantum level, both geometrical structures η and ω enter in the commutation relations of string operators. ω controls the non-commutativity of the zero-modes while η controls the non-commutativity of the string oscillator modes. This can be seen if one introduces a double notation for the string coordinate $\mathbb{X}(\sigma)$ that includes the string map X and its dual $\tilde{X}$. The string commutation relations, were derived in [7,8]

$$[\mathbb{X}^A(\sigma), \mathbb{X}^B(\sigma')] = 2i\lambda^2 \left[\pi\omega^{AB} - \eta^{AB}\theta(\sigma - \sigma')\right], \tag{20}$$

where $\theta(\sigma)$ is the staircase distribution, i.e., a solution of $\theta'(\sigma) = 2\pi\delta(\sigma)$; it is odd and quasi-periodic with period 2π.

Following standard practice, all indices are raised and lowered using η and η^{-1}. The momentum density operator is given by $\mathbb{P}_A(\sigma) = \frac{1}{2\pi\alpha'}\eta_{AB}\partial_\sigma\mathbb{X}^B(\sigma)$ and the previous commutation relation implies that it is conjugate to $\mathbb{X}^A(\sigma)$. The two-form ω appears when one integrates this canonical commutation relation to include the zero-modes, the integration constant being uniquely determined by worldsheet causality. Denoting by $(\hat{\mathbb{X}}, \hat{\mathbb{P}})$ the zero mode components of the string operators $\mathbb{X}(\sigma)$ and $\mathbb{P}(\sigma)$ we simply have that

$$[\hat{\mathbb{P}}_A, \hat{\mathbb{P}}_B] = 0, [\hat{\mathbb{X}}^A, \hat{\mathbb{P}}_B] = i\hbar\delta^A{}_B, [\hat{\mathbb{X}}^A, \hat{\mathbb{X}}^B] = 2\pi i\lambda^2\omega^{AB}. \tag{21}$$

This is a deformation of the doubled Heisenberg algebra involving the string length λ as a deformation parameter.

So far we have assumed that the background is trivial, with the fields (η, ω) *constant* and given by $\eta(\mathbb{K}, \mathbb{K}') = k \cdot \tilde{k}' + \tilde{k} \cdot k'$, and $\omega(\mathbb{K}, \mathbb{K}') = k \cdot \tilde{k}' - \tilde{k} \cdot k'$. As shown in [7], we can turn on non-trivial backgrounds encoded into ω by changing the $\mathrm{O}(d,d)$ frame $\mathbb{X} \to O\mathbb{X}$. This change of frame preserves η but transforms ω. Any constant ω can be obtained this way. Since ω has an interpretation as the symplectic form on the space of

$\mathbb{X}$'s, modifying ω affects the commutation relations[5] $[\hat{\mathbb{X}}^A, \hat{\mathbb{X}}^B] = 2\pi i\lambda^2\Pi^{AB}$,, with $\Pi^{AB}\omega_{BC} = \delta^A{}_C$, where we have introduced the Poisson tensor $\Pi = \omega^{-1}$.

For instance, under a constant B-field transformation $\mathbb{X} = (x^a, \tilde{x}_a) \mapsto (x^a, \tilde{x}_a + B_{ab}x^b)$, the trivial symplectic form $\omega(\mathbb{K}, \mathbb{K}') = k \cdot \tilde{k}' - \tilde{k} \cdot k'$ is mapped onto $\omega(\mathbb{K}, \mathbb{K}') = k_a\tilde{k}'^a - k'_a\tilde{k}^a - 2B_{ab}\tilde{k}^a\tilde{k}'^b$, and the commutators read

$$[\hat{x}^a, \hat{x}^b] = 0, \qquad [\hat{x}^a, \hat{\tilde{x}}_b] = 2\pi i\lambda^2\delta^a{}_b, \qquad [\hat{\tilde{x}}_a, \hat{\tilde{x}}_b] = -4\pi i\lambda^2 B_{ab}. \tag{22}$$

We see that the effect of the B-field is to render the dual coordinates non-commutative (and that the B-field originates from the symplectic structure ω). More generally, we can parameterize an arbitrary $O(d,d)$ transformation as $g = e^{\hat{B}}\hat{A}e^{\hat{\beta}}$, where $\hat{A} \in GL(d)$ and $e^{\hat{B}} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$ and $e^{\hat{\beta}} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ are nilpotent. $e^{\hat{B}}$ is the B-field transformation discussed above, and is associated with the usual B-field deformation in string theory. We note that the transformation of $(x^a, \tilde{x}_a)$ given above does not modify x^a, and thus fields that depend only on x^a are unmodified. The β-transformation on the other hand corresponds to the map $(x^a, \tilde{x}_a) \mapsto (x^a + \beta^{ab}\tilde{x}_b, \tilde{x}_a)$. Equivalently, it has the effect of mapping the symplectic structure to $\omega(\mathbb{K}, \mathbb{K}') = k_a\tilde{k}'^a - k'_a\tilde{k}^a + 2\beta^{ab}k_a k'_b$, and yields commutation relations

$$[\hat{x}^a, \hat{x}^b] = 4\pi i\lambda^2\beta^{ab}, \qquad [\hat{x}^a, \hat{\tilde{x}}_b] = 2\pi i\lambda^2\delta^a{}_b, \qquad [\hat{\tilde{x}}_a, \hat{\tilde{x}}_b] = 0. \tag{23}$$

Dramatically, the coordinates that are usually thought of as the spacetime coordinates have become themselves non-commutative. Since this is the result of an $O(d,d)$ transformation, we know that it can be thought of in similar terms as the B-field; these are related by T-duality. We are familiar with the B-field background because we have, in the non-compact case, a fixed notion of locality in the target space theory. However, in the non-geometric β-field background, we do not have such a notion of locality but we can access it through T-duality.

4. Effective quantum fields and manifest non-locality

What is the effective description of closed strings that incorporates the above intrinsic non-commutativity? For a closed sting on a circle of radius R (where the dual radius $\tilde{R}$, is defined as $R\tilde{R} = 2\lambda^2$ and the respective winding integers are n and w) this effective description is captured by the generalized field [7, 8]

$$\Phi(x, \tilde{x}) \equiv \sum_w \Phi_w(x)e^{iw\tilde{x}/\tilde{R}}. \tag{24}$$

[5] The algebraic structure that we are working with here has an analogy in electromagnetism in the presence of monopoles. In that analogy, the string length becomes the magnetic length, and the form ω becomes the magnetic field. Another analogy occurs in quantum Hall liquids, the algebra being the magnetic algebra of the lowest Landau level.

This meshes well with the observation [7, 8] that the string product is essentially a representation of the Heisenberg group, which suggests that one should consider the "quantization" map

$$\Phi(x,\tilde{x}) \to \hat{\Phi} = \sum_{w} \Phi_w(\hat{x}) e^{iw\hat{\tilde{x}}/\tilde{R}}, \tag{25}$$

from generalized fields to non-commutative fields.[6] Under this map the T-duality transformation becomes "localized" and is expressed as the exchange of $\hat{x}$ with $\hat{\tilde{x}}$. The T-dual expression is given by [7, 8]

$$\hat{\Phi} = \sum_{n} e^{in\hat{x}/R} \Phi_n(\hat{\tilde{x}} - \pi n\tilde{R}) = \sum_{n} \Phi_n(\hat{\tilde{x}}) e^{in\hat{x}/R}, \tag{26}$$

which has a similar form to (25). We see that the non-commutativity of $\hat{x}$ with $\hat{\tilde{x}}$ allows one to reabsorb all the shifts in terms of a simple reordering that exchanges $\hat{x}$ with $\hat{\tilde{x}}$ and is the expression of T-duality. The "quantized" field is simply expanded in terms of modes as

$$\hat{\Phi} \equiv \sum_{w,n} e^{in\hat{x}/R} \Phi(n,w) e^{iw\hat{\tilde{x}}/\tilde{R}}. \tag{27}$$

It is useful at this point to generalize the construction to higher dimensional tori. This can be done in a straightforward manner by introducing the modes $\mathbb{K}^A = (\tilde{k}^a, k_a)$, generalizing $(w/\tilde{R}, n/R)$. The integrality condition for the lattice Λ of admissible modes $\mathbb{K}, \mathbb{K}' \in \Lambda$ reads in this notation as[7] $(\eta \pm \omega)(\lambda\mathbb{K}, \lambda\mathbb{K}') \in \mathbb{Z}$. We now write $\Phi(\mathbb{K}) = \langle \mathbb{K}|\Phi\rangle$ with the ordering chosen as $\langle \mathbb{K}| = \langle 0|\hat{U}_{-\mathbb{K}}$, where $\hat{U}_{\mathbb{K}} \equiv e^{ik\cdot\hat{x}} e^{i\tilde{k}\cdot\hat{\tilde{x}}}$. This ordering can be seen to be related to the choice of an $\mathrm{O}(d,d)$ frame, where we place the operator associated with x on the left and the operator associated with the dual space $\tilde{x}$ on the right. The key point is that this choice of frame is entirely encoded into the choice of symplectic potential ω and the vertex operator can be covariantly written in terms of $\mathbb{K} = (\tilde{k}, k)$ and $\mathbb{X} = (x, \tilde{x})$ as

$$\hat{U}_{\mathbb{K}} = e^{\frac{i}{2}(\eta+\omega)(\mathbb{K},\hat{\mathbb{X}})} e^{\frac{i}{2}(\eta-\omega)(\mathbb{K},\hat{\mathbb{X}})}. \tag{28}$$

Given this notation we can write the string product covariantly as [7, 8]

$$(\Phi \circ \Psi)(\mathbb{K}) \;=\; \sum_{\mathbb{K}'+\mathbb{K}''=\mathbb{K}} \Phi(\mathbb{K}') e^{i\pi(\eta-\omega)(\lambda\mathbb{K}',\lambda\mathbb{K}'')} \Psi(\mathbb{K}''). \tag{29}$$

The non-commutativity of the string product is encoded in terms of a π-flux due to ω. As it turns out the phase factor is exactly the same as the cocycle factor $\epsilon(\mathbb{K},\mathbb{K}') = e^{i\pi(\eta-\omega)(\lambda\mathbb{K},\lambda\mathbb{K}')}$ that appears in the definition of the vertex operator product [7, 8].

[6] Here, we have chosen a specific operator ordering. Given this ordering, the mapping is well-defined and consistent with the string product.

[7] In the one dimensional case where $\mathbb{K} = (w/\tilde{R}, n/R)$ this follows directly from $(\eta + \omega)(\lambda\mathbb{K}, \lambda\mathbb{K}') = nw'$ and similarly $(\eta - \omega)(\lambda\mathbb{K}, \lambda\mathbb{K}') = wn'$, given that $n, n', w, w' \in \mathbb{Z}$.

4.1. Non-local excitations

To summarize, the above manifestly T-duality covariant formulation of closed strings (i.e. the metastring) implies intrinsic non-commutativity of zero-modes. It is thus instructive to formulate a particle-like limit of the metastring that we call the *metaparticle*. Given the form for the symplectic structure of the zero modes derived in section 4. of [7] (equation (67) of that paper, without the contribution coming from string oscillators), the action $S \equiv \int d\tau L$ of the metaparticle is governed by the following Lagrangian L

$$L = p_\mu \dot{x}^\mu + \tilde{p}^\mu \dot{\tilde{x}}_\mu - \pi\alpha' p_\mu \dot{\tilde{p}}^\mu - \frac{e}{2}\left(p_\mu p^\mu + \tilde{p}_\mu \tilde{p}^\mu - m^2\right) + \tilde{e}\left(p_\mu \tilde{p}^\mu - \mu^2\right). \quad (30)$$

where e and $\tilde{e}$ are the Lagrange multipliers for the two constraints that follow from the Hamiltonian ($H \equiv \partial_\sigma \mathbb{X}^A H_{AB} \partial_\sigma \mathbb{X}^B = 0$) and diffeomorphism constraints ($D \equiv \partial_\sigma \mathbb{X}^A \eta_{AB} \partial_\sigma \mathbb{X}^B = 0$) of the metastring [3, 6]. Note that the usual particle limit is obtained, at least classically, by taking $\mu \to 0$ and $\tilde{p} \to 0$. The theory of metaparticles can be viewed as the theory of the zero modes of the closed string, which fully takes into account its intrinsic non-commutativity. Given the form of the above Lagrangian, the metaparticle looks like two particles that are entangled through a Berry phase-like $p_\mu \dot{\tilde{p}}^\mu$ factor. The metaparticle is fundamentally non-local, and thus it should not be associated with effective local field theory. In particular, by looking at the metaparticle constraints $p^2 + \tilde{p}^2 = m^2$ and $p\tilde{p} = \mu^2$, we note that the momenta p and $\tilde{p}$ can be, in principle, widely separated. For example, if m is of the order of the Planck energy, and μ of the order of one TeV (which could be understood as a characteristic particle physics scale), then the momentum p can be of the order of the Planck energy, and the momentum $\tilde{p}$ of the vacuum energy scale! Thus the metaparticle theory is able to naturally relate widely separated scales, which transcends the usual reasoning based on Wilsonian effective field theory (and should be relevant for the naturalness and hierarchy problems).

We expect that the correct field theoretic description of the metaparticle is in terms of the above non-commutative (modular) field theory $\Phi(x, \tilde{x})$ limit of the metastring. Like their particle cousins, metaparticles should be detectable, and they might even present good candidates for dark matter quanta [46–51]. Such effective non-commutative field theory (similar in spirit to [52–54]) can be also useful in illuminating the vacuum energy problem, via the sequester mechanism [55] in which the dimensional parameters in the effective action of the matter sector depend on the four-volume element of the universe. Finally, we note that the concept of metaparticles might be argued from the compatibility of the quantum spacetime that underlies the generic representations of quantum theory, as discussed in [5], and thus the metaparticle might be as ubiquitous as the concept of antiparticles which is demanded by the compatibility of relativity and quantum theory.

5. Conclusions

In this talk we have discussed intrinsic non-commutativity in quantum gravity [7, 8] related to a new concept of quantum spacetime, called modular spacetime [4,5] that also appears as a habitat for metastring theory [3,6] and that is deeply rooted in the foundations of quantum theory. Note that this concept stems from a quantization of spacetime, and not from quantization of gravitational field/metric. Even the flat space is quantized according to our approach to quantum gravity. This allows for superposition and entanglement of spacetimes. Also, this formulation provides for an explicit construction of spacetime quanta or qubits, and a new non-perturbative definition of quantum gravity as "gravitization of the quantum" [1, 2].

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