# Hyperspherical three-body variables applied to lattice QCD Data*

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Abstract

Is the three-quark confinement potential better described by the Y-string or the $\Delta$-string form? In order to answer that question we have re-analysed the recent lattice QCD calculations by Koma & Koma [1] and the (much) older results of Takahashi et al. [2] using hyperspherical three-body variables. The (presently) extant lattice data do not allow a conclusive answer to the above question, but we show that hyperspherical coordinates lead to: 1) a definitive criterion for the discrimination between the Y and $\Delta$ strings; and 2) a more discriminating choice of lattice points to be made; both of these points are based on the dynamical O(2) symmetry of the Y-string.

## 1. Introduction

Lattice QCD offers a method to calculate three-quark potentials *ab initio*. The calculations are limited by the lattice size as well as by the computing- and man-power available. The results are therefore subject to systematic and statistical errors which must be estimated and interpreted as in any experimental result.

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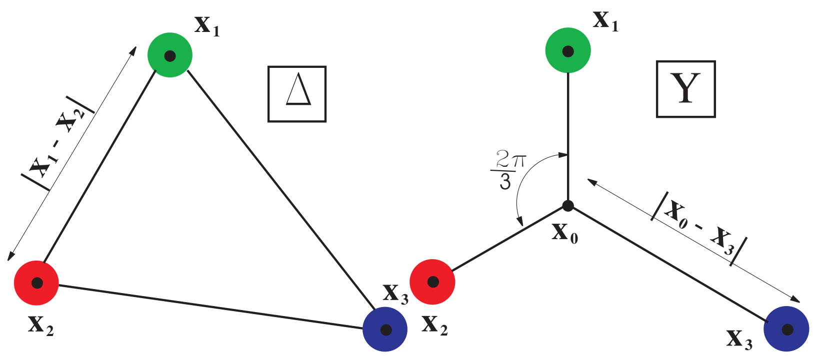


Figure 1: The $\Delta$ and Y- strings. The $\Delta$-string potential goes as the perimeter (sum of sides) of the triangle subtended by the three quarks, whereas the Y-string potential goes as the sum of lengths from each quark to the Fermat-Torricelli point of the three quark system.

The form of the three-heavy-quark potential in lattice QCD is not well known. Two older calculations of the effective static three-quark potential on the lattice are the one by Alexandrou et al [3], who proposed the $\Delta$-string form, Fig. 1, and the one by Takahashi et al. [2], who proposed the Y-string, Fig. 1. In the Y-string description, the potential depends on the sum of distances from the three quarks to the Fermat-Torricelli point (at the centre) of the system. In the $\Delta$-string description, the potential goes as the perimeter (the sum of sides) of the triangle subtended by the three quarks.

Explicit formulae for the functional forms of the Y and $\Delta$ string potentials are given in Refs. [4, 5, 6]. Both potentials depend linearly on the "overall size" variable, the hyper-radius $R$ (see Eq. (4) below) of the three-quark system. One way to distinguish the Y-string from the $\Delta$-string is to note that the Y-string has an O(2) dynamical symmetry [4, 5, 6]. This dynamical O(2) symmetry is visible to the naked eye when using permutation-adapted hyper-spherical coordinates, see Fig. 2 in [5]. The dynamical O(2) symmetry ought to be used in any conclusive attempt to distinguish between the Y-string and the $\Delta$-string potential.

The recent calculation of the three-quark potential by Koma & Koma [1], on a larger ($24^4$) lattice than before, has re-ignited interest in this question. These authors did not test the potential for the O(2) dynamical symmetry, however, so their work cannot be conclusive with regard to the Y vs. $\Delta$ dilemma.

The aim of this work is to use hyper-spherical coordinates to re-analyse both the Koma and the Takahashi data so as to see if it is possible on the basis of presently available lattice data to decide on the Y vs. $\Delta$ dilemma. We shall show that a clear resolution is not possible, due to an unfortunate choice of triangle shapes.

## 2. Hyper-spherical coordinates

The three bodies' Cartesian coordinates can be expressed in terms of the centre-of-mass variable $\mathbf{R}_{CM}$, and the two relative Jacobi vectors $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$.

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x_1} - \mathbf{x_2}), \tag{1}$$

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x_1} + \mathbf{x_2} - 2\mathbf{x_3}), \tag{2}$$

$$\mathbf{R}_{\mathrm{CM}} = \frac{1}{3}(\mathbf{x_1} + \mathbf{x_2} + \mathbf{x_3}), \tag{3}$$

which obscures the permutation symmetry, however. The confining potential must be invariant under: 1) translations - the potential must not depend on $\mathbf{R}_{CM}$, but only on the relative Jacobi vectors; 2) rotations - the potential can only depend on the scalar products of $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$. 3) Permutations of particle labels. There is one preferred set of hyper-angles that makes the permutation symmetry manifest. The hyper-radius $R$

$$R^2 = \boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2 \tag{4}$$

determines, loosely speaking, the size of the triangle subtended by the three particles. The hyper-angles $\alpha$ and $\phi$, which describe the shape of the triangle, are defined as follows

$$\phi = \arctan\left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}\right) \tag{5}$$

$$\alpha = \arccos\left(\frac{2(\boldsymbol{\rho}\times\boldsymbol{\lambda})}{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}\right) \tag{6}$$

The two hyper-angles determine a point on a unit-radius shape sphere. Figure 1 in Ref. [6] shows a view of the shape sphere from infinity above the North Pole.

## 3. Lattice QCD data in terms of hyper-spherical coordinates

The data from the papers was converted to our symmetric hyper-spherical coordinates by the equations described in section 2.. All of the data points are taken from the surface of a single hemisphere of the shape sphere. We plotted the data points calculated by Takahashi and Koma & Koma in the $x$-$y$ projection of the shape sphere. Figure 2 includes all permutations of the three-body system. There are three lines representing isosceles triangles - that cross the origin. There are also three lines representing the right-angled triangles - one such line is at $y = -0.5$. Equilateral triangles are located at the center of the circle (origin of the coordinate system). Collinear configurations all lie on a circle of radius 1 about the origin.

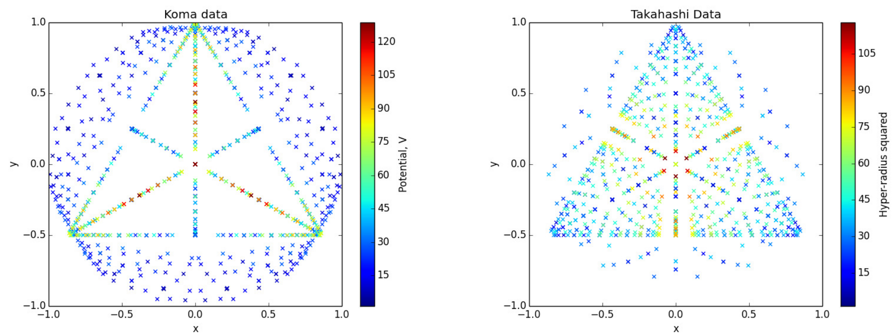


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Figure 2: (a): Hyper-radius as a function of hyper-angles for Koma & Koma data points. (b): Hyper-radius as a function of hyper-angles for Takahashi data points.

Note that the two data sets contain strikingly different shapes: whereas most of the Koma points (triangle shapes) lie outside of the straight lines (triangle) defined by the right-triangle shapes, most of the Takahashi points lie inside of this boundary. Note moreover, that most of the Takahashi points that lie inside the right-triangle lines are low-hyper-radii ones (blue to greenish color - see the colour code), which makes this data set unsuitable to address the Y vs. Δ dilemma, as one needs high(er) values of the hyper-radius, so as to suppress the Coulomb term which QCD generates in addition to the confining part.

## 4. Analysis of lattice data

Following standard lattice QCD treatises, Refs. [7], [8], we assume that the total three-quark potential $V_{3q}$ has the form

$$V_{3q}(\alpha, \phi, R) = -\frac{A(\alpha, \phi)}{R} + B(\alpha, \phi)R + C, \tag{7}$$

henceforth referred to as the Coulomb + linear potential Ansatz. The first term represents the sum of QCD Coulomb pairwise interactions, which is dominant at small values of the hyper-radius $R$. The second term represents the confinement potential, which is linear in $R$ and dominant at large values of hyper-radius $R$, and the third term - $C$ - is a constant. Here $A(\phi, \alpha)$ is assumed to be the (standard) sum of of pair-wise Coulomb terms, and $B(\phi, \alpha)$ is the unknown hyper-angular dependence of the linearly rising confining potential.

Our initial goal was to determine the unknown hyper-angular dependence $B(\phi, \alpha)$ of the linearly rising confining part of the three-quark potential using the lattice data and the well-known hyper-angular and hyper-radial dependences of the two-body Coulomb term.

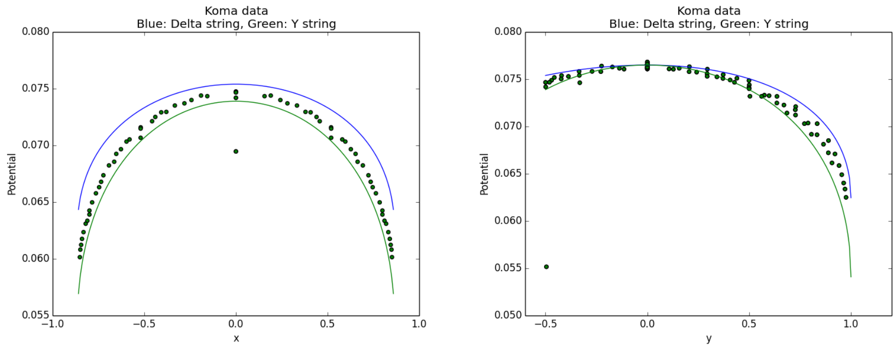


Figure 3: Plot of the hyperangular part of confining potential as a function of $x$ or $y$, for isosceles and right-angled triangle configurations in the Koma & Koma data set - blue line is the $\Delta$ string ad the green line is the Y string prediction. (l.h.s. panel): Right-angled triangles; (r.h.s. panel): Isosceles triangles.

Surprisingly, however, Koma & Koma [1] have noticed that the first term - the sum of pairwise Coulomb terms - does not adequately describe the lattice data as the hyper-radius $R$ decreases: the discrepancy amounts up to 26 % of the total Coulomb potential, depending on the shape of the triangle (see Eq. (34) in Sect. III.E "The functional form of the three-quark potential"). This indicates existence of a non-negligible three-body dependence of the QCD Coulomb potential that is not predicted by perturbative QCD calculations. This fact/circumstance additionally complicates our analysis and attempts to extract $B(\phi, \alpha)$.

In order to minimize the influence of the Coulomb term, and of the constant $C$[1], we have used only lattice configurations that are either (a) far away from the two-body collision points, where the Coulomb term rises to infinity, on the shape sphere; or (b) have large values of the hyper-radius $R$, where the Coulomb term is suppressed as compared with the confining term. There are only a few such sets of such configurations in the Koma and the Takahashi data sets: (i) the isosceles; (ii) the right-angled triangles. We shall use them both.

(i) It can be seen in Fig. 3.b that for the isosceles triangle configurations $\phi = const.$ in the Koma & Koma data set, the $B(\alpha)$ values form a (more or less continuous) line in between the $\Delta$ and Y-string potentials' functional forms. This result supports the conclusions of the Koma & Koma [1] that the three-quark potential is neither pure Y nor pure $\Delta$ string. The Takahashi data set does not fit well with either functional form and is strongly scattered.

[1] Of course, there is the possibility that the "constant" $C$ is actually a function of the shape-sphere angles $C(\phi, \alpha)$.

(ii) The right-angled triangle configurations provide little insight: The functional forms for the Y and Δ string potentials are almost identical and do not allow clear separation. The Koma & Koma data points follow a similar shape in Fig. 3.a, but could be attributed to either functional form. In the Takahashi data set the points are too scattered to draw any conclusion.

## 5. Conclusions

We have analysed the lattice QCD data, Refs. [2, 1] in terms of permutation-adapted hyperspherical variables with a view towards resolving the Δ vs. Y string dilemma.

The analysis along the lines of isosceles triangles gives no conclusive answer to the question of the confinement potential form, in agreement with Ref. [1]. The Takahashi data, Ref. [2], is too scattered, while the Koma & Koma data, Ref. [1], suggests some mixture of the Y and Δ string. Similarly, the values of the potential for the right-angled triangle configurations provide little insight into the Δ vs. Y string dilemma.

This calls for new lattice calculations, where: 1) there is a constant hyper-radius; and 2) one of the hyper-angles is held constant while varying the other.

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