Symmetries of a bosonic string*

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Abstract

We consider the string theory of a closed and an open string and search for the transformations of the space-time fields which do not change the physical content of the theory, i.e. the symmetry transformations. In the open string theory we start with a modified action which has an additional surface term which enables the invariance of the complete action to the general coordinate transformations and the gauge transformations. The string theory is conformally invariant world sheet field theory. Therefore, the physics is preserved if the conformal field theories corresponding to the initial and the transformed field configurations are isomorphic. We show that the general coordinate transformations are T-dual to the gauge transformations.

1. Bosonic string actions

The dynamics of the bosonic string, moving in a curved background associated with the massless bosonic fields [1], a metric field $G_{\mu\nu} = G_{\nu\mu}$, a Kalb-Ramond field $B_{\mu\nu} = -B_{\nu\mu}$ and a dilaton field Φ is described by the string sigma model action. The background fields in which the string moves have to satisfy the space-time equations of motion

$$
\begin{aligned}
&R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} + 2 D_\mu \partial_\nu \Phi = 0, \\
&D_\rho B^\rho{}_{\mu\nu} - 2\partial_\rho \Phi B^\rho{}_{\mu\nu} = 0, \\
&4(\partial\Phi)^2 - 4 D_\mu \partial^\mu \Phi + \frac{1}{12} B_{\mu\nu\rho} B^{\mu\nu\rho} + 4\pi\kappa(D-26)/3 - R = 0, \quad (1)
\end{aligned}
$$

in order for the conformal invariance of the quantum theory to be preserved. Here $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$,

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$R_{\mu\nu}$ is a Ricci tensor, and D_μ is a covariant derivative with respect to the space-time metric. We broadly investigated the string sigma model for the solution of the space-time equations of motion called *a weakly curved background* [2] (of the first and second order which differ by the term in brackets), given by

$$G_{\mu\nu}(x) = g_{\mu\nu} + \left\{3h^2_{\mu\nu}(x)\right\},$$
$$B_{\mu\nu}(x) = b_{\mu\nu} + h_{\mu\nu}(x), \quad h_{\mu\nu}(x) = \frac{1}{3}B_{\mu\nu\rho}x^\rho. \tag{2}$$

For the open bosonic string we searched for the solution of the boundary conditions, for which we obtained the effective theory describing a closed effective string and investigated the commutativity of the effective string coordinates. We obtained that noncommutativity exists along the entire string when the Kalb-Ramond field is coordinate dependent.

For the closed string moving in a weakly curved background we developed the T-dualization procedure, based on the standard procedure, giving a prescription how to find theories T-dual to a given theory. The generalization of our procedure was made for the weakly curved background of the second order, which does not posses the global shift symmetry. Our procedure enabled T-dualization of an arbitrary space-time coordinate. For the first order weakly curved background we performed the T-dualization of the arbitrary set of the initial coordinates [3]. If we choose d coordinates and mark the T-dualizations performed along them, along the rest of the coordinates and along all coordinates by

$$\mathcal{T}^a = \circ_{n=1}^{d} T^{\mu_n}, \quad \mathcal{T}^i = \circ_{n=d+1}^{D} T^{\mu_n}, \quad \mathcal{T} = \circ_{n=1}^{D} T^{\mu_n}, \tag{3}$$

and analogously mark the T-dualizations in the dual space

$$\mathcal{T}_a = \circ_{n=1}^{d} T_{\mu_n}, \quad \mathcal{T}_i = \circ_{n=d+1}^{D} T_{\mu_n}, \quad \widetilde{\mathcal{T}} = \circ_{n=1}^{D} T_{\mu_n}, \tag{4}$$

then we obtain a set of string sigma models connected by the diagram

$$\begin{tikzcd}
 & S[x^i, y_a] \arrow[dl, shift left, "\mathcal{T}_a"] \arrow[dr, shift left, "\mathcal{T}^i"] & \\
S[x^\mu] \arrow[ur, shift left, "\mathcal{T}^a"] \arrow[rr, shift left, "\mathcal{T}"] & & S[y_\mu]. \arrow[ul, shift left, "\mathcal{T}_i"] \arrow[ll, shift left, "\widetilde{\mathcal{T}}"]
\end{tikzcd}$$

In a left vertex is the initial action, the bosonic string action in a conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$,

$$S[x] = \kappa \int_\Sigma d^2\xi \; \partial_+ x^\mu \Pi_{+\mu\nu}(x)\partial_- x^\nu, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma, \tag{5}$$

with the background field composition

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x).$$

The totally T-dualized theory [4], in the right vertex of the T-duality diagram is

$$^{\star}S[y] = \frac{\kappa^2}{2} \int d^2\xi \; \partial_+ y_\mu \Theta_-^{\mu\nu}[\Delta V^{(0)}(y)] \partial_- y_\nu, \tag{6}$$

with the background

$$\Theta_\pm^{\mu\nu}[\Delta V] = -\frac{2}{\kappa}(G_E^{-1} \Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu}[\Delta V] \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}[\Delta V], \tag{7}$$

which is the inverse of the initial background field composition

$$\Theta_\pm^{\mu\nu} \Pi_{\mp\nu\rho} = \frac{1}{2\kappa} \delta_\rho^\mu, \tag{8}$$

and its argument is equal to

$$\Delta V^{(0)\mu}(y) = -\kappa \theta_0^{\mu\nu} \Delta y_\nu^{(0)} + (g^{-1})^{\mu\nu} \Delta \tilde{y}_\nu^{(0)}, \tag{9}$$

where the tilde coordinates represent the flat space-time T-duals of the corresponding T-dual coordinates.

The central action is a partially T-dualized action, obtained applying the T-dualization procedure to coordinates x^a, given by

$$\begin{aligned} S[x^i, y_a] = \kappa \int d^2\xi \Big[& \partial_+ x^i \overline{\Pi}_{+ij}(x^i, \Delta V^a(x^i, y_a)) \partial_- x^j \\ & -\kappa \, \partial_+ x^i \Pi_{+ia}(x^i, \Delta V^a(x^i, y_a)) \widetilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a)) \partial_- y_b \\ & +\kappa \, \partial_+ y_a \widetilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a)) \Pi_{+bi}(x^i, \Delta V^a(x^i, y_a)) \partial_- x^i \\ & +\frac{\kappa}{2} \, \partial_+ y_a \widetilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a)) \partial_- y_b \Big]. \end{aligned} \tag{10}$$

The new background field compositions are defined in terms of the initial field composition restricted to corresponding subspaces and their inverses and the inverses of the totally T-dualized field compositions restricted to corresponding subspaces

$$\widetilde{\Theta}_\pm^{ab} \Pi_{\mp bc} = \Pi_{\mp cb} \widetilde{\Theta}_\pm^{ba} = \frac{1}{2\kappa} \delta_c^a, \tag{11}$$

and

$$\overline{\Pi}_{\pm ij} \Theta_\mp^{jk} = \Theta_\mp^{kj} \overline{\Pi}_{\pm ji} = \frac{1}{2\kappa} \delta_i^k. \tag{12}$$

The argument of the background fields is defined by

$$\Delta V^{(0)a}(x^i, y_a) = -\kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} + \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta x^{(0)i}$$
$$-\kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} - \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta\tilde{x}^{(0)i}$$
$$-\frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} + \tilde{\Theta}^{ab}_{0-}\Big]\Delta y^{(0)}_b - \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} - \tilde{\Theta}^{ab}_{0-}\Big]\Delta\tilde{y}^{(0)}_b,$$

where the tilde coordinates represent the flat space-time T-duals of the appropriate initial or T-dual coordinate. Despite the fact the T-dualization procedure was applied to x^a, there appear the T-duals of the undualized directions.

If the same procedure is applied to the effective theory of the open bosonic string, one can investigate the open string T-duality [5]. There are four relevant theories in this context. The initial open string theory

$$S \;=\; \kappa\int_\Sigma d^2\xi\,\partial_+ x^\mu \Pi_{+\mu\nu}(x)\partial_- x^\nu, \tag{13}$$

its effective closed string theory, obtained for the solution of the boundary conditions

$$S^{eff} \;=\; \kappa\int_{\Sigma^\star} d^2\xi\,\partial_+ q^\mu\,\Pi^{eff}_{+\mu\nu}(q, 2b\tilde{q})\,\partial_- q^\nu, \tag{14}$$

the T-dual of the effective theory

$$^\star\mathcal{S}^{eff} \;=\; \frac{\kappa^2}{2}\int_{\Sigma^\star} d^2\xi\,\partial_+\varrho_\mu(\Theta^{eff}_-)^{\mu\nu}(g_E^{-1}\tilde{\varrho}, 2bg_E^{-1}\varrho)\partial_-\varrho_\nu, \tag{15}$$

and the open string theory having its effective theory equal to this T-dual theory

$$\tilde{S} \;=\; \kappa\int_\Sigma d^2\xi\,\partial_+ y_\mu \widetilde{\Pi}^{\mu\nu}_+(y)\partial_- y_\nu.$$

The effective theory background $\Pi^{eff}_{\pm\mu\nu}$ is composed of the effective metric $G^{eff}_{\mu\nu}(q) = G^E_{\mu\nu}(q) = G_{\mu\nu} - 4B_{\mu\rho}(q)(G^{-1})^{\rho\sigma}B_{\sigma\nu}(q)$ and the effective Kalb-Ramond field $B^{eff}_{\mu\nu}(2b\tilde{q}) = -\frac{\kappa}{2}[g\Delta\theta(2b\tilde{q})g]_{\mu\nu}$, where $\Delta\theta$ is an infinitesimal part of the noncommutativity parameter for the initial coordinates. Its T-dual background $(\Theta^{eff}_\mp)^{\mu\nu}$ is just its inverse. They are both defined in the doubled spaces, given in terms of the appropriate coordinates and their doubles. The relation between these background fields resembles the relation of the closed string initial field composition and its T-dual. However, in that case T-dualization transformed a geometrical space into the double space. In the open string T-duality, this change is not present.

The T-dual space remains the geometrical space as the initial space and the T-dual background fields keep the same form as the initial open string background. In the open string case, the important role in the relation between the T-dual backgrounds plays a matrix C, which is introduced to define the connection between the variables of the open string theory T-dual and the effective theory T-dual.

2. The symmetries

So, we found a number of physically equivalent string sigma models. Enough to rise again the old question of what is the symmetry transformation of fields. In a classical theory it is the change in fields which does not change the classical action. If the string theory is defined by the σ-model action, the space-time fields appear in this world-sheet action as the coupling constants. So, the standard technique for obtaining the symmetries is not applicable for finding the symmetry transformation of these fields. One should instead consider the conformal field theories corresponding to different field configurations [7]. In order for the transformation of field to be a symmetry (which is in fact the symmetry of the space-time action) the change in fields should correspond to change in energy-momentum tensors of two isomorphic conformal field theories.

Let us therefore consider the energy-momentum tensor

$$T_\pm = \mp\frac{1}{4\kappa}(G^{-1})^{\mu\nu} j_{\pm\mu} j_{\pm\nu}, \tag{16}$$

given in terms of currents $j_{\pm\mu} = \pi_\mu + 2\kappa\Pi_{\pm\mu\nu}(x)x'^\nu$. The hamiltonian corresponding to the initial string sigma model lagrangian is

$$\mathcal{H}_C = T_- - T_+. \tag{17}$$

The space-time equations of motion for the background fields (1), come from the condition that the energy-momentum tensor on a quantum level should satisfy the Virasoro algebras

$$\begin{aligned}
\Big[\hat{T}_\pm(\varphi(\sigma)), \hat{T}_\pm(\varphi(\bar{\sigma}))\Big] &= i\hbar\Big[\hat{T}_\pm(\varphi(\sigma)) + \hat{T}_\pm(\varphi(\bar{\sigma}))\Big]\delta'(\sigma-\bar{\sigma}), \\
\Big[\hat{T}_\pm(\varphi(\sigma)), \hat{T}_\mp(\varphi(\bar{\sigma}))\Big] &= 0.
\end{aligned} \tag{18}$$

It is known that the similarity transformation $\hat{T}_\pm \to e^{-i\hat{\Gamma}}\hat{T}_\pm e^{i\hat{\Gamma}}$, which causes the transformation of energy-momentum tensor

$$\delta\hat{T}_\pm(\varphi) = -i\Big[\hat{\Gamma}, \hat{T}(\varphi)\Big], \tag{19}$$

preserves the Virasoro algebra.

Let us consider the classical theory and suppose the coordinates and momenta satisfy the standard Poisson brackets

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu \delta(\sigma - \bar\sigma), \tag{20}$$

consequently the Poisson brackets of currents are

$$\begin{aligned}
\{j_{\pm\mu}(\sigma), j_{\pm\nu}(\bar\sigma)\} &= \pm 2\kappa \Gamma_{\mp\mu,\nu\rho}\, x'^\rho(\sigma)\delta(\sigma-\bar\sigma) \pm 2\kappa G_{\mu\nu}(x(\sigma))\delta'(\sigma-\bar\sigma), \\
\{j_{\pm\mu}(\sigma), j_{\mp\nu}(\bar\sigma)\} &= \pm 2\kappa \Gamma_{\mp\rho,\mu\nu}\, x'^\rho(\sigma)\delta(\sigma-\bar\sigma),
\end{aligned} \tag{21}$$

where the generalized connection is given in terms of the Christoffel symbol $\Gamma_{\mu,\nu\rho} = \frac{1}{2}(\partial_\nu G_{\rho\mu} + \partial_\rho G_{\mu\nu} - \partial_\mu G_{\nu\rho})$ and the field strength of the field $B_{\mu\nu}$, $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ by

$$\Gamma_{\pm\mu,\nu\rho} = \Gamma_{\mu,\nu\rho} \pm B_{\mu\nu\rho}. \tag{22}$$

One can further calculate the Poisson brackets between the energy-momentum tensor and currents

$$\begin{aligned}
\{T_\pm(\sigma), j_{\pm\mu}(\bar\sigma)\} &= \pm\frac{1}{2\kappa}\Gamma_{\mp\rho,\mu\nu} j^\nu_\mp j^\rho_\pm \delta(\sigma-\bar\sigma) - j_{\pm\mu}(\sigma)\delta'(\sigma-\bar\sigma), \\
\{T_\pm(\sigma), j_{\mp\mu}(\bar\sigma)\} &= \pm\frac{1}{2\kappa}\Gamma_{\mp\rho,\nu\mu} j^\nu_\pm j^\rho_\mp \delta(\sigma-\bar\sigma),
\end{aligned} \tag{23}$$

to finally obtain that $T_\pm$ satisfy the Virasoro algebra

$$\begin{aligned}
&\{T_\pm(\sigma), T_\pm(\bar\sigma)\} = -(T_\pm(\sigma) + T_\pm(\bar\sigma))\delta'(\sigma-\bar\sigma), \\
&\{T_\pm(\sigma), T_\mp(\bar\sigma)\} = 0.
\end{aligned} \tag{24}$$

The classical analogue of transformation (19) is just

$$\delta T_\pm(\varphi) = \Big\{\Gamma, T(\varphi)\Big\}. \tag{25}$$

Let us demonstrate that the known symmetry of the string theory, a gauge transformation of the Kalb-Ramond field

$$\begin{aligned}
&\delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \\
&\delta_\Lambda G_{\mu\nu} = 0,
\end{aligned} \tag{26}$$

given in terms of the vector gauge parameters Λ_μ, causes the transformation of the energy-momentum tensor which can be expressed in terms of the generator of the symmetry Γ as in (25). The change in $G_{\mu\nu}$ and $B_{\mu\nu}$ causes the following transformation of $T_\pm$

$$\delta T_\pm = \frac{1}{2\kappa}\Big(\frac{1}{\kappa}\delta_\Lambda B_{\mu\nu} \pm \frac{1}{2}\delta_\Lambda G_{\mu\nu}\Big) j^\mu_\pm j^\nu_\mp. \tag{27}$$

We consider a following expression

$$\Gamma_\Lambda = 2\int_{-\pi}^{\pi} d\sigma \Lambda_\mu x'^\mu = \frac{1}{\kappa}\int_{-\pi}^{\pi} d\sigma \Lambda_\mu (j_+^\mu - j_-^\mu), \tag{28}$$

and calculate its bracket with the energy-momentum tensor

$$\{\Gamma_\Lambda, T_\pm(\sigma)\} = \frac{1}{2\kappa^2}(\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu) j_\pm^\mu j_\mp^\nu. \tag{29}$$

Comparing (29) and (27), keeping in mind (26), we conclude that Γ_Λ is the generator of this symmetry.

2.1. The closed string symmetry and generators

Now, let us start the other way around. Let us suppose the background fields undergo a small change in value

$$\Pi_{\pm\mu\nu} \to \Pi_{\pm\mu\nu} + \delta\Pi_{\pm\mu\nu}. \tag{30}$$

This causes the following change of currents

$$\delta j_{\pm\mu} = 2\kappa \delta\Pi_{\pm\mu\nu}(x) x'^\nu, \tag{31}$$

and consequently the energy-momentum tensor $T_\pm = \mp\frac{1}{4\kappa}(G^{-1})^{\mu\nu} j_{\pm\mu} j_{\pm\nu}$ changes by

$$\delta T_\pm = \frac{1}{2\kappa}\delta\Pi_{\pm\mu\nu} j_\pm^\mu j_\mp^\nu. \tag{32}$$

If we demand this change to be equal to the change in the energy momentum tensor (25), we will obtained the corresponding symmetry transformation laws and their generators.

We assume the form of the generator of the general transformation (32), is the same as in (28), and therefore consider the expression $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ with

$$\mathcal{G}_\pm = \int d\sigma\, \Lambda_\pm^\mu(x(\sigma)) j_{\pm\mu}(\sigma). \tag{33}$$

The Poisson brackets between energy-momentum tensor and these quantities are

$$\begin{aligned}\{T_\pm(\sigma), \mathcal{G}_\pm(\bar\sigma)\} &= \pm\frac{1}{2\kappa}\Big(D_{\mp\nu}\Lambda_\pm^\mu\Big) j_\mp^\nu j_{\pm\mu},\\ \{T_\pm(\sigma), \mathcal{G}_\mp(\bar\sigma)\} &= \pm\frac{1}{2\kappa}\Big(D_{\pm\nu}\Lambda_\mp^\mu\Big) j_\pm^\nu j_{\mp\mu},\end{aligned} \tag{34}$$

where the covariant derivatives are given by $D_{\pm\mu}\Lambda^\nu = \partial_\mu \Lambda^\nu + \Gamma^\nu_{\pm\rho\mu}\Lambda^\rho = D_\mu \Lambda^\nu \pm B^\nu{}_{\rho\mu}\Lambda^\rho$. One demands the energy-momentum tensor transforms only by a similarity transformation, and therefore the change (32) is in fact

$$\delta T_\pm = \{\mathcal{G}, T_\pm\} = \frac{1}{2\kappa}\delta\Pi_{\pm\mu\nu} j_\pm^\mu j_\mp^\nu.$$

This law determines the closed string symmetry transformations, allowed by a similarity transformation. Separating the parameters in the generators (33) into $\Lambda_{\pm\mu} = \xi_\mu \pm \Lambda_\mu$, one obtains that the symmetry transformation of fields are

$$\begin{aligned} \delta G_{\mu\nu} &= -2(D_\mu \xi_\nu + D_\nu \xi_\mu), \\ \delta B_{\mu\nu} &= D_\mu \Lambda_\nu - D_\nu \Lambda_\mu - 2B_{\mu\nu}{}^\rho \xi_\rho. \end{aligned} \tag{35}$$

The generator $\mathcal{G}$ of these transformations can be expressed as

$$\mathcal{G} = \int d\sigma \Big[2\xi^\mu \pi_\mu + 2\kappa \widetilde{\Lambda}_\mu x'^\mu \Big]. \tag{36}$$

with

$$\widetilde{\Lambda}_\nu = 2\xi^\mu B_{\mu\nu} + \Lambda^\mu G_{\mu\nu} = \Lambda_\nu - 2B_{\nu\mu}\xi^\mu. \tag{37}$$

The form of the generator makes it T-dual invariant, which will be discussed later.

3. The open string and its symmetries

From (35) and (37), we can conclude that the closed string is invariant under the general coordinate transformations

$$\begin{aligned} \delta_\xi G_{\mu\nu} &= -2(D_\mu \xi_\nu + D_\nu \xi_\mu), \\ \delta_\xi B_{\mu\nu} &= -2\xi^\rho B_{\rho\mu\nu} + 2(\partial_\mu b_\nu - \partial_\nu b_\mu), \quad b_\mu = B_{\mu\nu}\xi^\nu, \end{aligned} \tag{38}$$

with $D_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho_{\mu\nu}\xi_\rho$, and the local gauge transformations

$$\begin{aligned} &\delta_\Lambda G_{\mu\nu} = 0, \\ &\delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \end{aligned} \tag{39}$$

where we omit tilde in lambda ($\tilde{\Lambda}_\mu \to \Lambda_\mu$). These transformations are T-dual to each other, as can be seen comparing their generators. The action is invariant under these transformations in the closed string case, but is not invariant in the open string case. In paper [6], the open string action was proposed, which is invariant to the above transformations because of the additional surface term which is added to the standard action.

For the open string, beside the equations of motion the minimal action principle gives the boundary conditions on the string end-points

$$\begin{aligned} &\gamma_0^\mu \delta x^\mu \Big|_{\sigma=0,\pi} = 0, \\ &\gamma_0^\mu = x'^\mu - 2(G^{-1}B)^\mu{}_\nu \dot{x}^\nu. \end{aligned} \tag{40}$$

For each of the coordinates one can fulfill these conditions by choosing either the Neumann or the Dirichlet boundary condition. Let us mark the coordinates with the Neumann condition by x^a, $a = 0, 1, \cdots, p$ and the coordinates with the Dirichlet condition by x^i, $i = p+1, \cdots, D-1$.

The additional part of the open string action introduced in [6] is given in terms of the boundary conditions and it reads

$$S_{\partial\Sigma} = 2 \int d\tau \Big[\kappa A_\mu(x)\dot{x}^\mu - \bar{A}_\mu(x)(G^{-1})^{\mu\nu}\gamma_\nu^{(0)}\Big]\Big|_{\sigma=0}^{\sigma=\pi} . \tag{41}$$

This term makes the open string theory invariant, taken that the introduced vector fields A_μ and $\bar{A}_\mu$ transform as

$$\begin{aligned} \delta_\Lambda A_\mu &= -\Lambda_\mu, \\ \delta_\xi \bar{A}_\mu &= -\xi_\mu. \end{aligned} \tag{42}$$

When the choice how to satisfy the boundary conditions is made, the surface term (41) reduces to

$$S_{\partial\Sigma} = 2 \int d\tau \Big[\kappa A_a^N(x)\dot{x}^a - A_i^D(x)(G^{-1})^{ij}\gamma_j^{(0)}\Big]\Big|_{\sigma=0}^{\sigma=\pi} , \tag{43}$$

where A_a^N and A_i^D are $(p+1)$- and $(D-p-1)$-dimensional vector gauge fields, first living on the Dp-brane and the second orthogonal to Dp-brane.

It is well known that the Kalb-Ramond field term of the action

$$\int d^2\xi \varepsilon^{\alpha\beta} B_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu$$

can be regarded as an analogue of the coupling of the Maxwell field

$$\int d\tau A_\mu \dot{x}^\mu .$$

So, for the directions satisfying the Neumann boundary condition, the action on the boundary equals

$$S_{\partial\Sigma} = 2\kappa \int d\tau A_a^N(x)\dot{x}^a \Big|_{\sigma=0}^{\sigma=\pi} , \tag{44}$$

and it can be rewritten as

$$S = \kappa \int d^2\xi \, \mathcal{F}_{ab}^{(a)} \, \varepsilon^{\alpha\beta} \partial_\alpha x^a \partial_\beta x^b , \tag{45}$$

with

$$\mathcal{F}_{ab}^{(a)} = \partial_a A_b^N(x) - \partial_b A_a^N(x). \tag{46}$$

The complete surface term can be written in terms of the field strengths as

$$S_{\partial\Sigma} = \kappa \int d^2\xi \, \Big(\mathcal{F}_{\mu\nu}^{(a)} \, \varepsilon^{\alpha\beta} + \frac{1}{2}\mathcal{F}_{\mu\nu}^{(s)} \, \eta^{\alpha\beta}\Big)\partial_\alpha x^\mu \partial_\beta x^\nu , \tag{47}$$

with

$$\mathcal{F}^{(a)}_{\mu\nu} = \partial_\mu \mathcal{A}_{0\nu}(x) - \partial_\nu \mathcal{A}_{0\mu}(x),$$
$$\mathcal{F}^{(s)}_{\mu\nu} = 2(\partial_\mu \mathcal{A}_{1\nu}(x) + \partial_\nu \mathcal{A}_{1\mu}(x)), \tag{48}$$

where

$$\mathcal{A}_{0a} = A^N_a, \quad \mathcal{A}_{0i} = 2B_{ij}(G^{-1})^{jk} A^D_k, \tag{49}$$

and

$$\mathcal{A}_{1a} = 0, \quad \mathcal{A}_{1i} = -A^D_i. \tag{50}$$

Comparing the boundary actions (45) and (47) with the action (5), we conclude that the addition of the surface term has changed the background fields by

$$G_{\mu\nu} \to G_{\mu\nu} + \mathcal{F}^{(s)}_{\mu\nu} = \mathcal{G}_{\mu\nu}$$
$$B_{\mu\nu} \to B_{\mu\nu} + \mathcal{F}^{(a)}_{\mu\nu} = \mathcal{B}_{\mu\nu}. \tag{51}$$

3.1. The generators

Now that we have rewritten the open string action in the same form as the standard closed string action, we can determine the open string symmetry transformations and their generators in analogy to the closed string symmetry transformations and their generators. For the open string with Neumann boundary conditions the symmetry transformations are just (instead of (35))

$$\begin{aligned} \delta G_{\mu\nu} &= -2D_\mu \xi_\nu - 2D_\nu \xi_\mu, \\ \delta B_{\mu\nu} &= D_\mu \Lambda_\nu - D_\nu \Lambda_\mu + 2\mathcal{B}_\mu{}^\rho{}_\nu \xi_\rho, \end{aligned} \tag{52}$$

where $\mathcal{B}_\mu{}^\rho{}_\nu$ is a field strength of the changed Kalb-Ramond field $B_{\mu\nu} \to \mathcal{B}_{\mu\nu} = B_{\mu\nu} + \mathcal{F}^{(a)}_{\mu\nu}$, in comparison to the closed string case. However, the field strength of the additional part is zero and therefore $\mathcal{B}_{\mu\nu\rho} = B_{\mu\nu\rho}$. The generator of these transformations is

$$\mathcal{G} = \int d\sigma \Big[2\xi^\mu \pi_\mu + 2\kappa(2\xi^\mu \mathcal{B}_{\mu\nu} + \Lambda^\mu G_{\mu\nu}) x'^\nu\Big], \tag{53}$$

with

$$\pi_\mu = \kappa G_{\mu\nu}(x)\dot{x}^\nu - 2\kappa \mathcal{B}_{\mu\nu}(x) x'^\nu. \tag{54}$$

The open string symmetry transformations, for all the remaining choices for the boundary conditions are

$$\begin{aligned} \delta \mathcal{G}_{\mu\nu} &= -2(\mathbf{D}_\mu \xi_\nu + \mathbf{D}_\nu \xi_\mu), \\ \delta \mathcal{B}_{\mu\nu} &= \mathbf{D}_\mu \Lambda_\nu - \mathbf{D}_\nu \Lambda_\mu - 2\mathcal{B}_{\mu\nu}{}^\rho \xi_\rho, \end{aligned} \tag{55}$$

with the open string covariant derivative equal to

$$\mathbf{D}_\mu \Lambda^\nu = \partial_\mu \Lambda^\nu + \mathbf{\Gamma}^\nu_{\rho\mu} \Lambda^\rho,$$

where $\mathbf{\Gamma}^\nu_{\rho\mu}$ is a Christoffel symbol for the changed metric $\mathcal{G}_{\mu\nu} = G_{\mu\nu} + \mathcal{F}^{(s)}_{\mu\nu}$. Again the field strength of the changed Kalb-Ramond field $\mathcal{B}_{\mu\nu} = B_{\mu\nu} + \mathcal{F}^{(a)}_{\mu\nu}$ is just $\mathcal{B}_{\mu\nu\rho} = B_{\mu\nu\rho}$. The generator of the transformation is

$$\mathcal{G} = \int d\sigma \Big[2\xi^\mu \pi_\mu + 2\kappa \widetilde{\mathbf{\Lambda}}_\nu x'^\nu \Big], \tag{56}$$

with

$$\pi_\mu = \kappa \mathcal{G}_{\mu\nu}(x) \dot{x}^\nu - 2\kappa \mathcal{B}_{\mu\nu}(x) x'^\nu, \tag{57}$$

and

$$\widetilde{\mathbf{\Lambda}}_\nu = 2\xi^\mu \mathcal{B}_{\mu\nu} + \Lambda^\mu \mathcal{G}_{\mu\nu} = \Lambda_\nu - 2\mathcal{B}_{\nu\mu}\xi^\mu. \tag{58}$$

Using (48), we obtain the connection

$$\mathbf{\Gamma}_{\mu,\nu\rho} = \Gamma_{\mu,\nu\rho} - 2\partial_\nu \partial_\rho A^D_\mu. \tag{59}$$

So, if we chose A^D_μ linear in coordinate, the connection will remain the same as in the closed string case. Therefore, for such a choice, the symmetry transformations remain the same in all the cases considered. The generators however differ.

4. Conclusion

We considered the string theory of a closed and an open string, described by a standard string sigma model and the modified open string action introduced in [6]. We searched for the symmetry transformations of the space-time fields in which the strings move. We found these symmetries comparing the change in the energy-momentum tensor caused by a similarity transformation and the transformation of fields. These transformations are the symmetries because the conformal field theories corresponding to the initial and the transformed field configurations are isomorphic.

We obtained the explicit form of symmetry transformations for the closed string and the open string with the arbitrary choice of boundary conditions. It turned out that for the appropriate choice of the form of the vector fields on the boundary the symmetry transformations are the same in all cases considered. The generators of these transformations are of the following form

$$\mathcal{G} = 2 \int d\sigma \Big[\xi^\mu \pi_\mu + \widetilde{\Lambda}_\mu \kappa x'^\mu \Big] = \mathcal{G}_\xi + \mathcal{G}_\Lambda. \tag{60}$$

Let us at this point include the T-duality into the consideration, namely the complete T-dualization, i.e. the T-dualization along all initial coordinates.

The T-dualization procedure gives the T-duality coordinate transformation laws, which are connecting the coordinates of the initial string sigma model $S[x]$, defined in (5) and the totally T-dualized theory $^\star S[y]$, defined in (6). The coordinate transformation laws are obtained comparing the solutions of the equations of motion for the gauge fields of the auxiliary actions. They read

$$\partial_\pm x^\mu \cong -\kappa\, \Theta_\pm^{\mu\nu}[\Delta V^{(0)}]\partial_\pm y_\nu \mp 2\kappa \Theta_{0\pm}^{\mu\nu}\beta_\nu^\mp[V^{(0)}].$$

In the canonical form, in the zeroth order they reduce to

$$\pi_\mu \cong \kappa x'^\mu.$$

It is obvious, that in this case the components of the symmetry generator (60) are T-dual to each other

$$\mathcal{G}_\xi \cong \mathcal{G}_\Lambda.$$

Therefore, the generator of symmetries is T-dual to itself.

The broader presentation of these investigations will be presented elsewhere, as well as the consideration of the connection between generators in the more complicated backgrounds.

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