Power law method for finding soliton solutions of the $2D$ Ricci flow model

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Abstract

The paper reviews few general methods which are used for investigating the integrable models and for finding their analytic solutions.Two specific methods, the symmetry method and the auxiliary equation method, will be especially considered. They will be applied to the Ricci flow model expressed as a nonlinear PDE in $2+1$ dimensions. Both methods have a similar philosophy: replacing the model by an ODE obtained through similarity reduction (in the approach based on symmetry), respectively by passing to the wave variable. The focus will be on the auxiliary equation method and a new approach, called power law method will be propsed.

1. Introduction

An important question in the study of the nonlinear differential equations is related to their integrability, that is establishing if solutions of the considered equations exist. There is not a general theory/procedure allowing to completely solve nonlinear ODEs or PDEs. Sometimes it is quite enough to decide if the system is integrable or not. There are many methods which have been proposed for deciding on the integrability of the nonlinear equations such as: the inverse scattering method [1, 2], the Hirota bilinear transformation [3], the generalized Riccati equation method [4], the $\left(\frac{G'}{G}\right)$-expansion method [5, 6], the Lie symmetry method [7, 8, 9, 10, 11], the soliton ansatz method [12, 13], the generalized conditional symmetry approach [14, 15] and other techniques. In this paper we will focus on two such methods: the symmetry approach and the expansion method applied to a specific nonlinear PDE, namely the 2D Ricci flow equation [16], [17]. It has been used by mathematicians in order to understand special geometries which admit 3-manifolds. These mathematical developments have been useful in physics too. Many papers dealing with physical applications of the Ricci flow, among which [18], [19], [20], [21], have been published. In [22] the Lie point symmetries are calculated in terms of two arbitrary functions. Conservation laws for the $2D$ Ricci flow model via the direct construction method [23], have been established in [24] and some new invariant solutions have been derived.

The paper is organized as follow: after this introductory section, in Section 2 the symmetry method will be presented. Section 3 will review the auxiliary equation method, while in Section 4 the $2D$ Ricci flow model will be analyzed. As a novelty, a most general polynomial expansion which includes almost all the approaches usually applied in literature will be proposed.

2. Symmetries and their applications in nonlinear dynamics

Many natural phenomena are described by a system of nonlinear partial differential equations (PDEs) which is often difficult to be solved analytically, since there is no general theory for completely solving nonlinear pdes. One of the most useful techniques for finding exact solutions for the dynamical systems described by nonlinear pdes is *the symmetry method*. On the one hand, we can consider the symmetry reduction of differential equations and thus obtain classes of exact solutions. On the other hand, by definition, a symmetry transforms solutions into solutions, and thus symmetries can be used to generate new solutions from known ones. The classical Lie method [25], the nonclassical method [26], the direct method [27], the differential constraint approach [28], the new nonclassical algorithm, the generalized symmetry method [29] are important methods for finding special classes of solutions for nonlinear PDEs.

The **Lie symmetry method** has been proven to be a powerful tool for studying a remarkable number of PDEs arising in mathematical physics [30], [31].

Let us consider a dynamical system described by the n-th order partial differential equation:

$$\Delta_\nu(x, u^{(n)}[x]) = 0, \tag{1}$$

where $x \equiv \{x^i,\ i = \overline{1,p}\} \subset R^p$ represent independent variables, while $u \equiv \{u^\alpha, \alpha = \overline{1,q}\} \subset R^q$ dependent ones. The notation $u^{(n)}$designates the set of variables which includes u and the partial derivatives of u up to the n-th order.

The general infinitesimal symmetry operator has the form:

$$U = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{2}$$

The n-th extension of (2) is given by:

$$U^{(n)} = U + \sum_{\alpha=1}^{q} \sum_{J} \phi_{\alpha}^{J}(x, u^{(n)}) \frac{\partial}{\partial u_J^{\alpha}}, \tag{3}$$

where

$$u_J^{\alpha} = \frac{\partial^m u^{\alpha}}{\partial x^{j_1} \partial x^{j_2} .. \partial x^{j_m}}. \tag{4}$$

Also, in (4) the second summation refers to all the multi-indices $J = (j_1, ...j_m)$, with $1 \leq j_m \leq \ p, 1 \leq m \leq \ n$ .The coefficient functions ϕ_{α}^{J} are given by the following formula:

$$\phi_{\alpha}^{J}(x^i, u^{(n)}) = \mathcal{D}_J[\phi_{\alpha} - \sum_{i=1}^{p} \xi^i u_i^{\alpha}] + \sum_{i=1}^{p} \xi^i u_{J,i}^{\alpha},\ \alpha = \overline{1, q}, \tag{5}$$

in which

$$u_i^{\alpha} = \frac{\partial u^{\alpha}}{\partial x^i},\ i = \overline{1, p}, \tag{6}$$

$$u_{J,i}^{\alpha} = \frac{\partial u_J^{\alpha}}{\partial x^i} = \frac{\partial^{m+1} u^{\alpha}}{\partial x^i \partial x^{j_1} \partial x^{j_2} .. \partial x^{j_m}}, \tag{7}$$

$$\mathcal{D}_J = \mathcal{D}_{j_1} \mathcal{D}_{j_2} ... \mathcal{D}_{j_m} = \frac{d^{\ m}}{dx^{j_1} dx^{j_2} .. dx^{j_m}}. \tag{8}$$

The Lie symmetries represent the set of all the infinitesimal transformations which keep invariant the differential system. The invariance condition is:

$$U^{(n)}[\Delta_{\nu}] \mid_{\Delta_{\nu}=0} = 0. \tag{9}$$

The characteristic equations associated to the general symmetry generator (2) have the form:

$$\frac{dx^1}{\xi^1} = ... = \frac{dx^p}{\xi^p} = \frac{du^1}{\phi_1} = ... = \frac{du^q}{\phi_q}. \tag{10}$$

By integrating the characteristic system of ordinary differential equations (10), the invariants I_r, $r = \overline{1, (p + q - 1)}$ of the analyzed system can be found. They are identified the constants of integration. Following this way, the set of similarity variables is found in terms through which the original evolutionary equation with p independent variables and q dependent ones can be reduced to a set of differential equations with $(p + q - 1)$ variables. These are the similarity reduced equations which generate the similarity solution of the analyzed model.

The **nonclassical symmetry method** (NMS) may be used to derive nonclassical symmetries which are the one-parameter groups of transformations acting on the space of the independent and dependent variables of the systems that leave only a subset of the set of all analytical solutions invariant.

The basic idea of the nonclassical method is to augment (1) with the invariance surface condition:

$$\Omega_\alpha \equiv \phi_\alpha(x,u) - \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial u^\alpha}{\partial x^i} = 0,\ \ \alpha = \overline{1,q}. \tag{11}$$

The $q-$tuple $Q = (Q^1, Q^2, ...Q^q)$ is known as the characteristic of the symmetry operator (2).

The infinitesimal invariance of the equation (1) and the invariant surface condition (11) can be written:

$$U^{(n)}(\Delta_\nu)_{\Delta_\nu=0,\Omega_\alpha=0} = 0,\ \ U^{(n)}(\Omega_\alpha)_{\Delta_\nu=0,\Omega_\alpha=0} = 0. \tag{12}$$

By extending the conditions (12), we see that the number of determining equations for the infinitesimals $\xi^i(x,u)$, $\phi_\alpha(x,u)$ appearing in the nonclassical method is smaller than the one for the classical method. The main difficulty of this approach is that the determining equations are no longer linear. On the other hand, the NSM may produce more solutions than the CSM, since any classical symmetry is a nonclassical one, but conversely this does not apply [32].

For $\xi^p \neq 0$ in [33] is proposed a new algorithm for finding nonclassical symmetries. This algorithm is based on the following remark: since U is a symmetry operator so is αU, for any function $\alpha = \alpha(x,u)$. Thereby, if $\xi^p \neq 0$ we could multiplicate U by $(1/\xi^p)$ and write down the invariant surface condition under the equivalent expression:

$$\frac{\partial u}{\partial x^p} = \eta(x,u) - \sum_{i=1}^{p-1} \xi^i(x,u)\frac{\partial u}{\partial x^i}. \tag{13}$$

By substituting (13) and its derivatives with respect to x in (1) a new partial differential equation results:

$$\Delta' \equiv \Delta'(F_\beta(x,u), u^{[l]}, ..., u^{[n]}), \tag{14}$$

for the unknown function $u = u(x^1, ..., x^{p-1}; x^p)$ of $x^1, ..., x^{p-1}$ (here x^p is considered as a parameter); the functions $F_\beta(x,u)$ are the coefficients of $u^{[l]}$, where $u^{[l]}$ denotes the whole set of partial derivatives of u with respect to $x = (x^1, ..., x^{p-1})$ up to order N.

The next step consists in applying the classical Lie method to (14). Thus, let us consider a Lie group of point transformations associated with the following infinitesimal symmetry generator:

$$V = \sum_{i=1}^{p} s^i(x,u)\frac{\partial}{\partial x^i} + r(x,u)\frac{\partial}{\partial u}. \tag{15}$$

By imposing the invariance of (14) under the action of operator (15), the determining differential equations for the infinitesimals $s^i(x,u), r(x,u)$.will be obtained. By resetting the coefficient functions $s^p = 1$, $s^i = \xi^i$, $i = 1,...,p-2$, $r = \eta$ and substituting the functions F_β into the determining equations, we could find determining equations for nonclassical symmetries of the original dynamical system described by PDE of type (1).

3. The expansion method

The second method we are focusing on is the so called "expansion method". Its idea consists in reducing the governing PDE to an ODE. To do that, the first step to be followed is the introduction of the wave variable $\xi = \xi(t, x^1, ..., x^p)$. The second step is to look for solutions of the ODE in terms of the solutions of another ODE, called "auxiliary equation", with already known solutions.

Let us consider a governing equation of the form:

$$F(u, u_t, u_x, u_{xx}, u_{tt}, ...) = 0 \tag{16}$$

We define the wave coordinate:

$$\xi = x - Vt \tag{17}$$

By that, the equation (16) becomes the following ODE:

$$Q(u, u', u'', u''', ...) = 0 \tag{18}$$

where the derivatives are considered in respect with ξ.

There are many versions related to the expansion method, depending on the auxiliary equation which is chosen. Two of the most used versions are:

i)The tanh method - solutions of (18) in terms of $\tanh(\xi), \cosh(\xi), \sinh(\xi)$, etc. which are solutions $\phi(\xi)$ of equations, as Riccati, so:

$$u(\xi) = \sum_{i=0}^{N} a_i \varphi^i \tag{19}$$

ii) The G'/G method, where $G(\xi)$ solution of an auxiliary equation. In this case:

$$u(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'}{G}\right)^i \tag{20}$$

The approach we are proposing is an unifying one. More precisely, the solution of the master equation will be asked to be a polynomial expansion in terms of the solutions $G(\xi)$ of an "auxiliary equation":

$$u(\xi) = \sum_{i=0}^{N} P_i(G)(G')^i \tag{21}$$

where $P_i(G)$ are polynomials in G to be determined. Computing the derivatives of $u(\xi)$ higher order derivatives $G', G'', G\prime''$, ... could appear. So we might look to a more general solution depending on higher derivatives of $G(\xi)$:

$$u(\xi) = P_0(G) + P_1(G)G' + P_2(G, G')G'' + ... \tag{22}$$

Although, the higher derivatives $G''.G'''$,... can be expressed in terms of G, G' by using an adequate auxiliary equation. The form and the order of the auxiliary equations are important. The most used auxiliary equations which appear in the papers published in the last years are:

- Riccati Equation (first order nonlinear equation):

$$G' = \alpha + \beta G^2 \tag{23}$$

- Second order linear ODE:

$$G'' + AG' + BG = 0 \tag{24}$$

-Second order nonlinear ODE:

$$AGG'' - B(G')^2 - CGG' - EG^2 = 0 \tag{25}$$

-Third order nonlinear ODE:

$$AG^2G''' - B(G')^3 - CG(G')^2 - DG^2G' - FG^3 = 0 \tag{26}$$

The next step after choosing the auxiliary equation is to determine the limit N of the expansion (21) by a standard "balancing" procedure: replace (21) in (18) and take into account the higheast nonlinearity and the term with the maximal order of derivation.

In our case a new requirement is imposed: polynomial expantions for the functions $P_0(G), P_1(G), P_2(G)$,... To have a true balance and compatibility, we have to consider expansions of the form:

$$P_2(G) = \sum_{i=-2}^{0} a_i G^i \tag{27}$$

$$P_1(G) = \sum_{j=-1}^{0} b_j G^j \tag{28}$$

$$P_0(G) = c_0 \tag{29}$$

From the algebraic system generated by these choices, we can determine the coefficients a_i, b_i and c_i. After that, we can write down the form of the solutions $u(\xi)$. These solutions have to be discussed for various possible values of the coefficients A, B, C,...appearing in the master equation.

4. Application to the 2D Ricci flow equation

The 2D Ricci flow equation in has the form:

$$u_t = \frac{u_{xy}}{u} - \frac{u_x u_y}{u^2} \tag{30}$$

It was analyzed through symmetry method in [34].Here we will focus on the auxiliary equation method and its use in the direct finding of soliton type solutions. We will apply the wave transformation which transforms the equation (30) in:

$$U'U^2 + \frac{\alpha\beta}{\lambda}(UU'' - U'^2) = 0 \tag{31}$$

We will solve equation (31) by four different methods, in order to compare the solutions themselves and the efficiency of the methods. A general approach, unifying methods as tanh or G'/G, will be now proposed. It will be denominated as the *power law method*.

4.1. Double integration method

The equation (31) can be solved directly by double integration and the form of the solution is:

$$\begin{aligned} U &= \frac{e^A}{-1 + \lambda c_1 e^A} \\ A(\xi) &= \frac{\xi + c_2}{c_1 \alpha\beta} \end{aligned} \tag{32}$$

The direct integration leads to singular solution which are not of Physical interest.

4.2. Solution of tanh type

The simplest way of finding soliton solutions for (26) is to use Riccati as auxiliary equation and to look for solutions of (31) . More precisely, we will consider:

$$U(\xi) = \sum_{i=0}^{N} a_i G^i \tag{33}$$

The Riccati equation has the form:

$$G' = k + G^2 \tag{34}$$

with k a real constant.

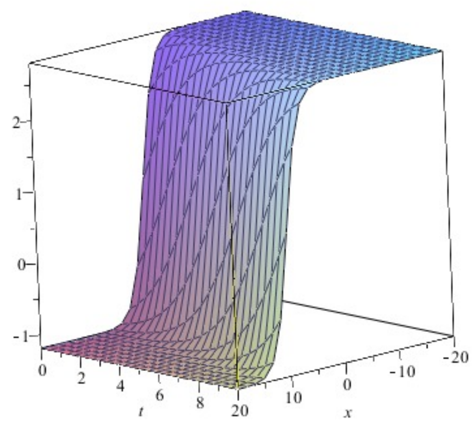

Figure 1: Tanh method $y = 0$

Considering integrating constant as zero,

$$
\begin{aligned}
&\xi = \left\{ \begin{matrix} \frac{1}{\sqrt{k}} \tan^{-1}\left(\frac{G}{\sqrt{k}}\right) \\ -\frac{1}{\sqrt{k}} \cot^{-1}\left(\frac{G}{\sqrt{k}}\right) \end{matrix} \right., k > 0 \\
&\xi = -\frac{1}{G}, k = 0 \\
&\xi = \left\{ \begin{matrix} -\frac{1}{\sqrt{-k}} \tanh^{-1}\left(\frac{G}{\sqrt{-k}}\right) \\ -\frac{1}{\sqrt{-k}} \coth^{-1}\left(\frac{G}{\sqrt{-k}}\right) \end{matrix} \right., k < 0
\end{aligned} \tag{35}
$$

The balancing procedure leads to the maximal value $N = 1$, that is the solution we are looking for will have the form:

$$U(\xi) = a_0 + a_1 G \tag{36}$$

$$U(\xi) = \frac{\alpha\beta}{\lambda}\left(\sqrt{-k} - \sqrt{-k}\tanh\sqrt{-k}\xi\right) \tag{37}$$

4.3. Solution of G'/G type

We solve now the equation by using the G'/G method. It imposes to look for solutions of the form:

$$U(\xi) = \sum_{i=0}^{N} d_i \left(\frac{G'}{G}\right)^i \tag{38}$$

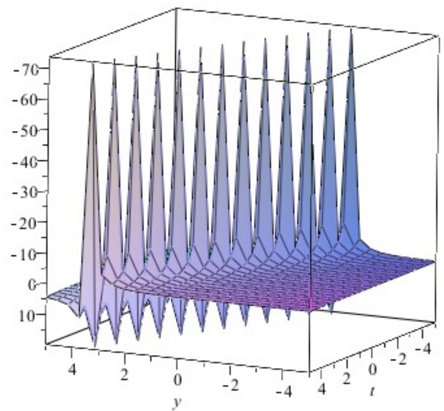

Figure 2: Polynomial expansion for $\alpha = 1, \beta = 2, \lambda = 1, x = 0$

We will consider that d_i are constant coefficients, while this time,$G(\xi)$ is a solution of the auxiliary equation of the form:

$$G'' + mG' + nG = 0 \tag{39}$$

Again, the balancing procedure leads to the same limit $N = 1$.

By introducing (38) in (39) we get a polynomial equation in G' containing monomyals until G'^7. Equating with zero the coefficients for all this monomials we get a system of 8 ODE with the unknown quantities $a_0(G), a_1(G)$ which satisfy the following equations:

$$a_0 = \frac{\alpha\beta m}{a_1} + Ce^{-\frac{a_1\lambda}{\alpha\beta}G} \tag{40}$$

$$a_1' a_1 + \frac{\alpha\beta}{\lambda}(a_1'' a_1 - a_1') = 0 \tag{41}$$

4.4. Power law expansion

Another quite similar approach for solving the "master" equation (31) consists in looking for polynomial solutions of the type:

$$U' = V(U) \tag{42}$$

Then:

$$\alpha\beta U V(U)\frac{\mathrm{d}V}{\mathrm{d}U} + \lambda U^2 V(U) - \alpha\beta V(U)^2 = 0. \tag{43}$$

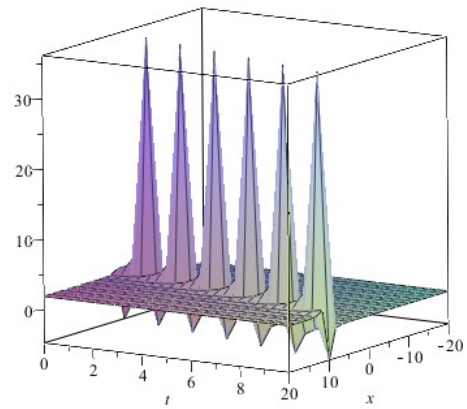

Figure 3: Power law method for $\alpha = 1, \beta = 1, \lambda = 1, y = 0, k = 1/2$

The solution of (43) is

$$V(U) = -\frac{\lambda U(2U + k)}{2\alpha\beta}, \tag{44}$$

where k is the integration constant. From the above relation we can find that

$$U = \frac{k}{\sinh(\frac{\lambda k\zeta}{2\alpha\beta}) + \cosh(\frac{\lambda k\zeta}{2\alpha\beta}) - 2} \tag{45}$$

The final solution is quite similar with the ones got through previous approaches and it is presented in the figure below.

5. Conclusions

We proposed a general algorithm for finding solutions of nonlinear PDEs by using polynomial expansions in terms of auxiliary equations solutions. It includes all the methods proposed in literature, known as $\tanh, \cosh, \sinh, G'/G$, etc. The main idea is quite similar with what symmetry method offers: to reduce a complicated equation to a simpler one, to solve this last equation, and to transfer its solutions to the master (complicated) equation. We pointed out the importance of three main factors: - the choice of the auxiliary equation; - the choice of the form of solution; - the balancing procedure. For the specific model we tackled, we get that the polynomials from (21) have the form:

$$P_2(G) = a_2G^{-2} \tag{46}$$
$$P_1(G) = b_1G^{-1} \tag{47}$$
$$P_0(G) = 0 \tag{48}$$

It appears in a natural way that really the largest class of solutions can be expressed as (G'/G) expansions. Why this expansion was choosen in previous approaches was not at all clear. The method we proposed is purelly analytic and it open the doors for finding other solutions which do not belong to the class of (G'/G) class.

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